CHAPTER 12

Continuity and Differentiability

Chapter Highlights

Continuity, Discontinuity of a function, Geometrical meaning of continuity

CONTINUITY

In this chapter, the concept of a continuous function is defined in terms of limits. Most of the results in calculus are not true unless we are dealing with functions that are continuous. We may intuitively think of continuous functions as those functions whose graphs we can draw without lifting the pencil. A formal definition of continuity follows:

Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at an interior point $x = a$ of its domain if $\lim_{x \to a} f(x) = f(a)$. In other words a function $f(x)$ is said to be continuous at a point $x = a$ provided lefthand limit righthand limit and value of function are equal:

IMPORTANT POINTS

A function $f(x)$ is continuous at a point $x = a$ if

$$\lim_{h \to 0} f(a - h) = \lim_{h \to 0} f(a + h) = f(a)$$

Continuity of a Function on An Interval

Continuity on an Open Interval

A function $f(x)$ is said to be continuous on an open interval (a, b) if it is continuous at each point of (a, b).

Continuity on a Closed Interval

A function $f(x)$ is said to be continuous on a closed interval $[a, b]$ if

1. $f(x)$ is continuous from right at $x = a$, i.e., $\lim_{h \to 0} f(a + h) = f(a)$
2. $f(x)$ is continuous from left at $x = b$, i.e., $\lim_{h \to 0} f(b - h) = f(b)$
3. $f(x)$ is continuous at each point of the open interval (a, b).

Continuity at end points of an Interval

For continuity of $f(x)$ at the end points of an interval $[a, b]$, we must have

- $\lim_{h \to 0} f(a + h) = f(a)$ at $x = a$
- $\lim_{h \to 0} f(b - h) = f(b)$ at $x = b$

TRICK(S) FOR PROBLEM SOLVING

For continuity of $f(x)$ at the end points of an interval $[a, b]$, we must have

(i) $\lim_{h \to 0} f(a + h) = f(a)$ at $x = a$.

(ii) $\lim_{h \to 0} f(b - h) = f(b)$ at $x = b$.

DISCONTINUITY OF A FUNCTION

A function $f(x)$, which is not continuous at a point $x = a$, is said to be discontinuous at that point.

TRICK(S) FOR PROBLEM SOLVING

The discontinuity may arise due to any of the following situations:

(i) $\lim_{h \to 0} f(a-h) \neq \lim_{h \to 0} f(a+h)$, i.e., LHL and RHL exist, but are not equal.

(ii) $\lim_{h \to 0} f(a-h) = \lim_{h \to 0} f(a+h) \neq f(a)$, i.e., LHL and RHL exist and are equal, but are different from $f(a)$.

(iii) $f(a)$ is not defined.

(iv) Atleast one of the limits $\lim_{h \to 0} f(a-h)$ or $\lim_{h \to 0} f(a+h)$ does not exist or atleast one of these limits is ∞ or $-\infty$.

Some Useful Results on Continuous Functions

1. If f and g are continuous at $x = a$, then
   (A) $f + g$ is continuous at $x = a$.
   (B) $f - g$ is continuous at $x = a$.
   (C) fg is continuous at $x = a$.
   (D) f/g is continuous at $x = a$, provided $g(a) \neq 0$.
   (e) kf is continuous at $x = a$, where k is any real constant.
   (f) $[f(x)]^{m/n}$ is continuous at $x = a$, provided $[f(x)]^{m/n}$ is defined on an interval containing a, and m and n are integers.
2. If f is continuous at a and g is continuous at $f(a)$, then gof is continuous at a.
3. If f is continuous at $x = a$ and g is discontinuous at $x = a$, then $f + g$ and $f - g$ are discontinuous at $x = a$, whereas fg may be continuous at $x = a$.
4. If f is continuous at $x = a$ and $f(a) \neq 0$, then there exits an open interval $(a - \delta, a + \delta)$ such that $\forall\, x \in (a - \delta, a + \delta)$, $f(x)$ has the same sign as $f(a)$.
5. If f is a continuous function defined on $[a, b]$ such that $f(a) \cdot f(b) < 0$, then there exists atleast one solution of the equation $f(x) = 0$ in the open interval (a, b).
6. If f is a continuous function defined on $[a, b]$ and k is any real number between $f(a)$ and $f(b)$, then there exists atleast one solution of the equation $f(x) = k$ in the open interval (a, b).
7. If a function f is continuous on a closed interval $[a, b]$, then it is bounded on $[a, b]$ i.e., there exists real numbers k and K such that
$$k \leq f(x) \leq K \text{ for all } x \in [a, b]$$
8. Every polynomial is continuous at every point of the real line.
9. Every rational function is continuous at every point where its denominator is different from zero.
10. Logarithmic functions, Exponential functions, Trigonometric functions, Inverse circular functions and Absolute value functions are continuous in their domain.

Types of Discontinuity

Removable Discontinuity

If $\lim_{h \to 0} f(a-h)$ and $\lim_{h \to 0} f(a+h)$ exist and are equal, but are not equal to $f(a)$, then the function $f(x)$ is said to have a removable discontinuity at $x = a$. However, by suitably defining the function at $x = a$, $f(x)$ can be made continuous at $x = a$.

Discontinuity of the First Kind

If $\lim_{h \to 0} f(a-h)$ and $\lim_{h \to 0} f(a+h)$ exist but are not equal, then the function $f(x)$ is said to have a discontinuity of the first kind at $x = a$.

We also say that $f(x)$ has jump discountinuity at $x = a$. We define $\left| \lim_{x \to a^-} f(x) - \lim_{x \to a^+} f(x) \right|$ = jump of the function at $x = a$.

If $\lim_{h \to 0} f(a-h)$ exists but is not equal to $f(a)$, then the function $f(x)$ is said to have a discontinuity of the first kind from the left at $x = a$.

Similarly, if $\lim_{h \to 0} f(a+h)$ exists but is not equal to $f(a)$, then the function $f(x)$ is said to have a discontinuity of the first kind from the right at $x = a$.

Discontinuity of the Second Kind

If atleast one of the limits $\lim_{h \to 0} f(a-h)$ or $\lim_{h \to 0} f(a+h)$ does not exist or atleast one of these limits is ∞ or $-\infty$, then the function $f(x)$ is said to have a discontinuity of the second kind at $x = a$. We also say that $f(x)$ has an infinite discontinuity at $x = a$.

If $\lim_{h \to 0} f(a-h)$ does not exist or is equal to ∞ or $-\infty$, then the function $f(x)$ is said to have a discontinuity of the second kind from the left at $x = a$. Discontinuity of the second kind from the right is similarly defined.

Oscillating Discontinuity

$f(x)$ has oscillating discontinuity at $x = a$ if $\lim_{h \to 0}$ and $\lim_{h \to 0}$ lie in a certain range but do not approach to a definite value at $x = a$.

GEOMETRICAL MEANING OF CONTINUITY

1. A function $f(x)$ will be continuous at a point $x = a$, if there is no break or cut or gap in the graph of the function $y = f(x)$ at the point $[a, f(a)]$. Otherwise, it is discontinuous at that point.

2. A function $f(x)$ will be continuous on the closed interval $[a, b]$ if the graph of the function $y = f(x)$ is an unbroken line (curved or straight) from the point $[a, f(a)]$ to the point $[b, f(b)]$.

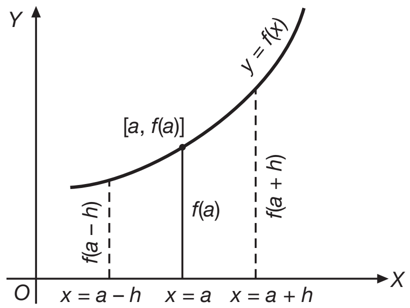


Fig. 12.1 $f(x)$ has a continuous graph at $x = a$

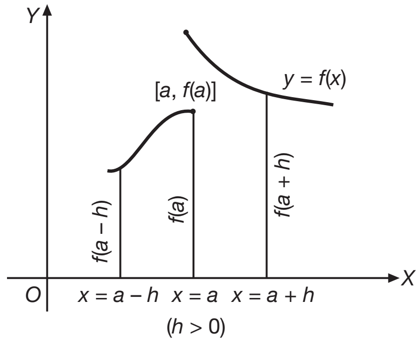


Fig. 12.2 $f(x)$ has a discontinuous graph at $x = a$

SOLVED EXAMPLES

1. If $f(x) = \begin{cases} \dfrac{e^{[x]+x} - 1}{[x] + x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$ then

(A) $\lim_{x \to 0^+} f(x) = -1$

(B) $\lim_{x \to 0^-} f(x) = \dfrac{1}{e} - 1$

(C) $f(x)$ is continuous at $x = 0$

(D) $f(x)$ is discontinuous at $x = 0$.

Solution: (D)

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} f(0 + h)$$

$$= \lim_{h \to 0} \frac{e^{(h)+h} - 1}{(h) + h}$$

$$= \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

and $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$

$$= \lim_{h \to 0} \frac{e^{(-h)-h} - 1}{(-h) - h}$$

$$= \lim_{h \to 0} \frac{e^{-1-h} - 1}{(-1-h)} = \frac{e^{-1} - 1}{-1}$$

$$= 1 - \frac{1}{e}$$

Since $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$, therefore, **$f(x)$ is not continuous at $x = 0$.**

2. If $f(x) = x \sin \dfrac{1}{x}$, $x \neq 0$, then the value of the function at $x = 0$, so that the function is continuous at $x = 0$, is

(A) 0 (B) – 1
(C) 1 (D) indeterminate

Solution: (A)

For $f(x)$ to be continuous at $x = 0$, we must have

$$f(0) = \lim_{x \to 0} f(x)$$

$$\Rightarrow \quad f(0) = \lim_{x \to 0} x \sin \frac{1}{x} = 0 \times a \text{ finite quantity} = 0$$

Hence $f(0) = 0$.

3. The function $f(x) = \begin{cases} (x+1)^{2-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ is

(A) discontinuous at only one point
(B) discontinuous exactly at two points
(C) continuous everywhere
(D) None of these

Solution: (A)

The only doubtful point is $x = 0$.

$$\text{L.H.L.} = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} (-h + 1)^{2-\left(\frac{1}{h}-\frac{1}{h}\right)}$$

$$= (1 - h)^2 = 1$$

$$\text{R.H.L.} = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} (h + 1)^{2-\left(\frac{1}{h}+\frac{1}{h}\right)}$$

$$= \lim_{h \to 0} (1 + h)^{2-\frac{2}{h}}$$

$$= \lim_{h \to 0} (1 + h)^2 [(1 + h)^{1/h}]^{-2}$$

$$= 1 \times e^{-2} = e^{-2}$$

Since LHL $\neq$ RHL,

$\therefore$ $f(x)$ is not continuous at $x = 0$.

4. If $f(x) = \dfrac{1}{1-x}$, then the points of discontinuity of the function $f[f\{f(x)\}]$ are

(A) $\{0, -1\}$ (B) $\{0, 1\}$
(C) $\{1, -1\}$ (D) None of these

Solution: (B)

$$\text{We have, } f(x) = \frac{1}{1-x}.$$

As at $x = 1$, $f(x)$ is not defined, $x = 1$ is a point of discontinuity of $f(x)$.

$$\text{If } x \neq 1, f[f(x)] = f\left(\frac{1}{1-x}\right) = \frac{1}{1 - 1/(1-x)} = \frac{x-1}{x}$$

$\therefore$ $x = 0, 1$ are points of discontinuity of $f[f(x)]$.

If $x \neq 0, x \neq 1$,

$$f[f\{f(x)\}] = f\left(\frac{x-1}{x}\right) = \frac{1}{1 - \frac{(x-1)}{x}} = x$$

5. If $f(x) = \begin{cases} \dfrac{e^{[x]+|x|} - 2}{[x]+|x|}, & x \neq 0 \\ -1 & x = 0 \end{cases}$, ([.] denotes the greatest integer function) then

(A) $f(x)$ is continuous at $x = 0$
(B) $\lim\limits_{x \to 0^+} f(x) = -1$
(C) $\lim\limits_{x \to 0^-} f(x) = 1$
(D) None of these

Solution: (D)

$$f(x) = \begin{cases} \dfrac{e^{[x]+|x|} - 2}{[x]+|x|}, & x \neq 0 \\ -1, & x = 0 \end{cases}$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{e^{[x]+|x|} - 2}{[x]+|x|} = \frac{e^{-1} - 2}{-1}$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{e^{[x]+|x|} - 2}{[x]+|x|}$$

$$= \lim_{x \to 0^+} \frac{e^{x} - 2}{x} \to -\infty$$

6. The Dirichlet function, defined as

$$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}, \text{ is}$$

(A) continuous for all real x
(B) continuous only at some values of x
(C) discontinuous for all real x
(D) discontinuous only at some values of x

Solution: (C)

Let x_0 be any arbitrary real number.

Case I: x_0 is rational

Then $f(x_0) = 1$

In any vicinity of a rational point there are irrational points, where $f(x) = 0$. Hence, in any vicinity of x_0 there are points x for which

$$|\Delta y| = |f(x_0) - f(x)| = 1$$

Case II: x_0 is irrational

Then $f(x_0) = 0$

In any vicinity of an irrational point there are rational points at which $f(x) = 1$. Hence, it is possible to find the values of x for which

$$|\Delta y| = |f(x_0) - f(x)| = 1$$

Thus, in both cases, the difference Δy does not tend to zero as $\Delta x \to 0$. Therefore, x_0 is a point of discontinuity. Since x_0 is an arbitrary point, the Dirichlet function **$f(x)$ is discontinuous at each point.**

7. If $f(x) = \begin{cases} x, \text{ when } x \text{ is rational} \\ 1 - x, \text{ when } x \text{ is irrational} \end{cases}$, then

(A) $f(x)$ is continuous for all real x
(B) $f(x)$ is discontinuous for all real x
(C) $f(x)$ is continuous only at $x = 1/2$
(D) $f(x)$ is discontinuous only at $x = 1/2$

Solution: (C)

Let a be any real number.

$$\lim_{x \to a} f(x) = \lim_{x \to a} x = a$$

(when $x \to a$ through rational values)

$$\lim_{x \to a} f(x) = \lim_{x \to a} (1 - x) = 1 - a$$

(when $x \to a$ through irrational values)

$\lim\limits_{x \to a} f(x)$ will exist only when $a = 1 - a$

i.e., when $a = \dfrac{1}{2}$.

Thus if $x \neq \dfrac{1}{2}$, then $\lim\limits_{x \to a} f(x)$ will not exist and hence $f(x)$ will be discontinuous at $x = a$ where $a \neq \dfrac{1}{2}$.

Also, $\lim_{x \to \frac{1}{2}} f(x) = \frac{1}{2}$ and $f\left(\frac{1}{2}\right) = \frac{1}{2}$

Hence, $f(x)$ is continuous at $x = \mathbf{\frac{1}{2}}$.

8. Let $f(x) = . \begin{cases} (1+|\sin x|)^{a/|\sin x|}, & \frac{-\pi}{6} < x < 0 \\ e^{\tan 2x/\tan 3x} & , 0 < x < \frac{\pi}{6} \\ b & , x = 0 \end{cases}$ The values of a and b so that $f(x)$ may be continuous at $x = 0$ are

(A) $a = \frac{-2}{3}, b = e^{2/3}$ (B) $a = \frac{2}{3}, b = e^{-2/3}$

(C) $a = \frac{2}{3}, b = e^{2/3}$ (D) None of these

Solution: (C)

We have,

$$\lim_{h \to 0} f(0-h) = \lim_{h \to 0} [1 + |\sin(-h)|]^{a/|\sin(-h)|}$$

$$= \lim_{h \to 0} (1 + \sin h)^{a/\sin h}$$

$$= \lim_{h \to 0} [(1 + \sin h)^{1/\sin h}]^a = e^a,$$

$$\lim_{h \to 0} f(0+h) = \lim_{h \to 0} e^{\tan 2h/\tan 3h}$$

$$= e^{\lim_{h \to 0}\left(\frac{\tan 2h}{2h} \times \frac{2}{3} \times \frac{3h}{\tan 3h}\right)} = e^{1 \times \frac{2}{3} \times 1} = e^{\frac{2}{3}}$$

and $f(0) = b$.

For f to be continuous at $x = 0$, we must have

$$\lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(0+h) = f(0) \Rightarrow e^a = e^{2/3} = b$$

$$\Rightarrow \quad a = \frac{2}{3} \text{ and } b = e^{2/3}$$

9. The function $f(x) = (\sin 2x)^{\tan^2 2x}$ is not defined at $x = \pi/4$. The value of $f(\pi/4)$ so that f is continuous at $x = \pi/4$ is

(A) $\sqrt{e}$ (B) $1/\sqrt{e}$

(C) 2 (D) None of these

Solution: (B)

f is continuous at $x = \pi/4$, if

$$\lim_{x \to \pi/4} f(x) = f(\pi/4).$$

Now, $$L = \lim_{x \to \pi/4} (\sin 2x)^{\tan^2 2x}$$

$$\Rightarrow \log L = \lim_{x \to \pi/4} \tan^2 2x \log \sin 2x$$

$$= \lim_{x \to \pi/4} \frac{\log \sin 2x}{\cot^2 2x} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \pi/4} \frac{2\cot 2x}{-2\cot 2x \operatorname{cosec}^2 2x \cdot 2} = -\frac{1}{2}$$

or $L = e^{-1/2} \therefore f(\pi/4) = e^{-1/2} = 1/\sqrt{e}$

10. Let a function $f: R \to R$ satisfy the equation $f(x+y) = f(x) + f(y)$ for all x, y. If the function $f(x)$ is continuous at $x = 0$, then

(A) $f(x) = 0$ for all x

(B) $f(x)$ is continuous for all positive real x

(C) $f(x)$ is continuous for all x

(D) None of these

Solution: (C)

Since $f(x)$ is continuous at $x = 0$,

$$\therefore \quad \lim_{x \to 0} f(x) = f(0)$$

Take any point $x = a$, then at $x = a$

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} [f(a) + f(h)]$$

$$[\because f(x+y) = f(x) + f(y)]$$

$$= f(a) + \lim_{h \to 0} f(h) = f(a) + f(0)$$

$$= f(a+0) = f(a)$$

$\therefore f(x)$ is continuous at $x = a$. Since $x = a$ is any arbitrary point, therefore $f(x)$ is continuous for all x.

11. If $f(x)$ is continuous in [0, 1] and $f\left(\frac{1}{2}\right) = 2$, then $\lim_{n \to \infty} f\left(\frac{\sqrt{n}}{2\sqrt{n}+1}\right)$ is equal to

(A) 0 (B) ∞

(C) 2 (D) None of these

Solution: (C)

Since $f(x)$ is continuous in [0, 1], therefore,

$$\lim_{n \to \infty} f\left(\frac{\sqrt{n}}{2\sqrt{n}+1}\right) = f\left(\lim_{n \to \infty} \frac{\sqrt{n}}{2\sqrt{n}+1}\right)$$

$$= f\left(\frac{1}{2}\right) = 2$$

12. The function $f(x) = (x)$, where (x) denotes the smallest integer $\geq x$, is

(A) continuous everywhere
(B) continuous at integral points only
(C) continuous at non-integral points only
(D) None of these

Solution: (C)

Let $x = n, n \in Z$

Then, L.H.L. $= \lim\limits_{\substack{x \to n \\ x < n}} (x) = n$; R.H.L. $= \lim\limits_{\substack{x \to n \\ x > n}} (x) = n + 1$

Since, L.H.L. ≠ R.H.L., therefore $f(x)$ is discontinuous at all integers n.

Now, let $x = p$, $n < p < n + 1$, where n is an integer.

Then, L.H.L. $= \lim\limits_{\substack{x \to p \\ x < p}} (x) = n + 1$, R.H.L.

$$= \lim_{\substack{x \to p \\ x > p}} (x) = n + 1$$

$$f(p) = (p) = n + 1$$

Since L.H.L. = R.H.L. = $f(p)$, therefore, **$f(x)$ is continuous at all non-integral points** p.

13. Let f be a function defined and continuous on [2, 5]. If $f(x)$ takes rational values for all x and $f(4) = 8$ then the value of $f(3.7)$ is

(A) 0 (B) 8
(C) – 1 (D) None of these

Solution : (B)

Since f is continuous on [2, 5], therefore f assumes atleast once, every values between $f(2)$ and $f(5)$. But it is given that $f(x)$ takes only rational values for all x and there are irrational values also between $f(2)$ and $f(5)$, this is possible only if $f(x)$ has a constant rational value at all points between $x = 2$ and $x = 5$. Since $f(4) = 8$, $\therefore$ **$f(3 \cdot 7) = 8$.**

14. Let $f(x) = [x^3 - 3]$, where [.] denotes the greatest integer function. Then the number of points in the interval (1, 2) where the function is discontinuous, is

(A) 4 (B) 2
(C) 6 (D) None of these

Solution: (C)

Let $g(x) = x^3 - 3$, then $g(x)$ is an increasing function on the interval (1, 2). Since $g(1) = -2$ and $g(2) = 5$, therefore between – 2 and 5 there are **6** points where $f(x)$ is discontinuous (as $[x^3 - 3]$ is discontinuous at the points where $x^3 - 3$ is an integer).

Differentiability of a Fucntion

The function, $f(x)$ is differentiable at a point P, iff there exists a unique tangent at the point P. In other words, $f(x)$ is differentiable at a point P iff the curve does not have P as a corner point, i.e., "the function is not differentiable at those points on which function has jumps (or holes) and sharp edges."

Consider the function $f(x) = |x|$. Let us draw the graph of this function

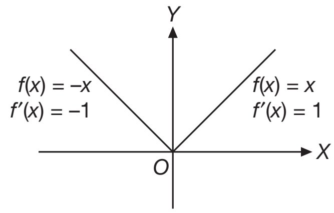


Fig. 12.3

which shows that $f(x)$ is not differentiable at $x = 0$ as $f(x)$ has sharp edge at $x = 0$.

Differentiability of a Function at a Point

Let f be a function defined on an interval $I \subseteq R$. We say that f is differentiable at an interior point $c \in I$ provided

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists and is finite.}$$

We denote this limit by $f'(c)$, called the *derivative of f at c*.

TRICK(S) FOR PROBLEM SOLVING

In view of the definition of limit of a function f, one may observe that $f'(c)$ exists provided

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

The limit on the left, denoted by $L\,f'(c)$, is called the *left hand derivative* of f at c and the limit on the right, denoted by $R\,f'(c)$, is called the *right hand derivative* of f at c.

Thus, $f(x)$ is differentiable at $x = c$ if $L\,f'(c) = R\,f'(c)$.

Differentiability of a Function on an Interval

A function $f(x)$ is said to be differentiable on an open interval (a, b) if $f(x)$ is differentiable at every point of this interval (a, b).

It is differentiable on a closed interval $[a, b]$ if it is differentiable on the open interval (a, b) and the limits.

$$R f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$$

and $$L f'(a) = \lim_{h \to 0} \frac{f(b-h) - f(b)}{(b-h) - b} \text{ exist.}$$

Some Important Results on Differentiability

1. Every polynomial function, exponential function and constant function is differentiable at each point of the real line.
2. Logarithmic functions, trigonometric functions and inverse trigonometric functions are differentiable in their domain.
3. The sum, difference, product and quotient of two differentiable functions is differentiable.
4. The composition of differentiable functions is a differentiable function.
5. If a function is not differentiable but is continuous at a point, it geometrically implies there is a sharp corner or a kink at that point.
6. If $f(x)$ and $g(x)$ both are not differentiable at a point, then the sum function $f(x) + g(x)$ and the product function $f(x) \times g(x)$ can still be differentiable at that point.

TRICK(S) FOR PROBLEM SOLVING

- If a function $f(x)$ is differentiable at a point $x = a$ then it is continuous at $x = a$.
- If $f(x)$ is only continuous at a point $x = a$, there is no guarantee that $f(x)$ is differentiable there.
- If $f(x)$ is not differentiable at $x = a$ then it may or may not be continuous at $x = a$.
- If $f(x)$ is not continuous at $x = a$, then it is not differentiable at $x = a$.
- If left hand derivative and right hand derivative of $f(x)$ at $x = a$ are finite (they may or may not be equal) then $f(x)$ is continuous at $x = a$.

SOLVED EXAMPLES

15. If $f(x) = \begin{cases} x[x], & 0 \le x < 2 \\ (x-1)[x], & 2 \le x \le 3 \end{cases}$, where [.] denotes the greatest integer function, then

(A) both $f'(1)$ and $f'(2)$ do not exist
(B) $f'(1)$ exists but $f'(2)$ does not exist
(C) $f'(2)$ exists but $f'(1)$ does not exist
(D) both $f'(1)$ and $f'(2)$ exist

Solution: (A)

We have, $$L f'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{(1-h)[1-h] - 1[1]}{-h}$$

$$= \lim_{h \to 0} \frac{0-1}{-h} \to \infty$$

$\therefore f'(1)$ does not exist.

Also, $$L f'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{(2-h)[2-h] - (2-1)[2]}{-h}$$

$$= \lim_{h \to 0} \frac{2-h-2}{-h} = 1$$

and $$R f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)[2+h] - (2-1)[2]}{h}$$

$$= \lim_{h \to 0} \frac{2+2h-2}{h} = 2.$$

$\therefore f'(2)$ also does not exist.

16. If $f(x) = \begin{cases} x^p \cos \frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$, then at $x = 0$, $f(x)$ is

(A) continuous if $p > 0$ and differentiable if $p > 1$
(B) continuous if $p > 1$ and differentiable if $p > 0$
(C) continuous and differentiable if $p > 0$
(D) None of these

Solution: (A)

Continuity at $x = 0$:

$$\text{L.H.L.} = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} (-h)^p \cos \frac{1}{h}$$

$$= 0 \text{ if } p > 0$$

$$\text{R.H.L.} = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} h^p \cos \frac{1}{h}$$

$$= 0 \text{ if } p > 0$$

and $f(0) = 0$.

$\therefore f(x)$ is continuous at $x = 0$ if $p > 0$

Differentiability at $x = 0$:

$$L f'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(-h)^p \cos \frac{1}{h} - 0}{-h}$$

$$= \lim_{h \to 0} (-h)^{p-1} \cos \frac{1}{h} = 0 \text{ if } p - 1 > 0,$$

i.e., $p > 1$;

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^p \cos \frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} h^{p-1} \cos \frac{1}{h} = 0 \text{ if } p > 1$$

$\therefore$ $f(x)$ is differentiable at $x = 0$ if $p > 1$.

17. Let $g(x) = x f(x)$, where $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

At $x = 0$,

(A) g is differentiable but g' is not continuous
(B) g is differentiable while f is not
(C) both f and g are differentiable
(D) g is differentiable and g' is continuous.

Solution: (A, B)

We have, $g(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

For $x \neq 0$,

$$g'(x) = x^2 \cos \frac{1}{x} \left(-\frac{1}{x^2}\right) + 2x \sin \frac{1}{x}$$

$$= -\cos \frac{1}{x} + 2x \sin \frac{1}{x}$$

For $x = 0$

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x} - 0}{x}$$

$$= \lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

$$\therefore \quad g'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

g' is not continuous at $x = 0$ as $\cos \frac{1}{x}$ is not continuous at $x = 0$. Also, **f is not differentiable at $x = 0$.**

18. If $f(x) = \begin{cases} \frac{1}{1 + e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is

(A) continuous as well as differentiable at $x = 0$
(B) continuous but not differentiable at $x = 0$
(C) differentiable but not continuous at $x = 0$
(D) None of these

Solution: (D)

$$\lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{1}{1 + e^{-1/h}} = \frac{1}{1+0} = 1$$

and $\lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{1}{1 + e^{1/h}} = \lim_{h \to 0} \frac{e^{-1/h}}{e^{-1/h} + 1}$

$$= \frac{0}{1} = 0$$

Since $\lim_{h \to 0} f(0-h) \neq \lim_{h \to 0} f(0+h)$, therefore, $f(x)$ is not continuous and hence not differentiable at $x = 0$.

19. If $f(x) = \begin{cases} \frac{e^{1/x} - 1}{e^{1/x} + 1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is

(A) continuous as well as differentiable at $x = 0$
(B) continuous but not differentiable at $x = 0$
(C) differentiable but not continuous at $x = 0$
(D) None of these

Solution: (D)

$$\lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1} = \frac{0-1}{0+1}$$

$$= -1$$

and $\lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1} = \lim_{h \to 0} \frac{1 - e^{-1/h}}{1 + e^{-1/h}}$

$$= \frac{1-0}{1+0} = 1$$

Since $\lim_{h \to 0} f(0-h) \neq \lim_{h \to 0} f(0+h)$, therefore, $f(x)$ is not continuous and hence not differentiable at $x = 0$.

20. The function $f(x) = \begin{cases} \frac{x(e^{1/x} - e^{-1/x})}{e^{1/x} + e^{-1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is

(A) continuous everywhere but not differentiable at $x = 0$
(B) continuous and differentiable everywhere
(C) not continuous at $x = 0$
(D) None of these

Solution: **(A)**

Differentiability at $x = 0$:

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-h\,(e^{-1/h} - e^{1/h})}{-h\,(e^{-1/h} + e^{1/h})}$$

$$= \lim_{h \to 0} \frac{e^{-2/h} - 1}{e^{-2/h} + 1} = -1$$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h\,(e^{1/h} - e^{-1/h})}{h\,(e^{1/h} + e^{-1/h})}$$

$$= \lim_{h \to 0} \frac{1 - e^{-2/h}}{1 + e^{-2/h}} = 1$$

Since $L f'(0) \neq R f'(0)$, $\therefore f(x)$ is not differentiable at $x = 0$.
But since $L f'(0)$ and $R f'(0)$ are finite, therefore $f(x)$ is continuous at $x = 0$.
Hence, **$f(x)$ is continuous every where but not differentiable at $x = 0$.**

21. If $f(x) = [x - 2]$, then

(A) $f'(2.5) = \frac{1}{2}$ and $f'(5) = 3$
(B) $f'(2.5) = 0$ and $f'(5) = 3$
(C) $f'(2.5) = 0$ and $f'(5)$ does not exist
(D) both $f'(2.5)$ and $f'(5)$ do not exist

Solution: (C)

We have

$$L f'(2\cdot5) = \lim_{h \to 0} \frac{f(2\cdot5 - h) - f(2\cdot5)}{-h}$$

$$= \lim_{h \to 0} \frac{(2\cdot5 - h - 2) - (2\cdot5 - 2)}{-h}$$

$$= \lim_{h \to 0} \frac{0}{-h} = 0$$

and $$R f'(2\cdot5) = \lim_{h \to 0} \frac{f(2\cdot5 + h) - f(2\cdot5)}{h}$$

$$= \lim_{h \to 0} \frac{(2\cdot5 + h - 2) - (2\cdot5 - 2)}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = 0$$

$\therefore \quad f'(2\cdot5) = 0$

Also, $$L f'(5) = \lim_{h \to 0} \frac{f(5-h) - f(5)}{-h}$$

$$= \lim_{h \to 0} \frac{(5 - h - 2) - (5 - 2)}{-h}$$

$$= \lim_{h \to 0} \frac{2 - 3}{-h} \to \infty$$

Hence, **$f'(2\cdot5) = 0$ while $f'(5)$ does not exist.**

22. If $f(x) = [\tan x]$, $x \in \left(0, \frac{\pi}{3}\right)$, then $f'\left(\frac{\pi}{4}\right)$ is equal to

(A) 1 (B) 0
(C) does not exist (D) None of these

Solution: (C)

$$L f'\left(\frac{\pi}{4}\right) = \lim_{h \to 0} \frac{f\left(\frac{\pi}{4} - h\right) - f\left(\frac{\pi}{4}\right)}{-h}$$

$$= \lim_{h \to 0} \frac{\left[\tan\left(\frac{\pi}{4} - h\right)\right] - \left[\tan\frac{\pi}{4}\right]}{-h}$$

$$= \lim_{h \to 0} \frac{0 - 1}{-h} \to \infty$$

$\therefore f'\left(\frac{\pi}{4}\right)$ does not exist.

23. If $\operatorname{sgn}(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then the function $f(x) = \operatorname{sgn}[\operatorname{sgn}(x)]$ is

(A) continuous as well as differentiable at $x = 0$
(B) continuous but not differentiable at $x = 0$
(C) differentiable but not continuous at $x = 0$
(D) neither differentiable nor continuous at $x = 0$

Solution: (D)

We have,

$$\operatorname{sgn}[\operatorname{sgn}(x)] = \begin{cases} \operatorname{sgn}\left(\frac{|x|}{x}\right) & , x \neq 0 \\ \operatorname{sgn}(0) & , x = 0 \end{cases}$$

$$= \begin{cases} \dfrac{\left|\frac{|x|}{x}\right|}{\frac{|x|}{x}} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$$\therefore \qquad f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Continuity at $x = 0$:

$$\text{L.H.L.} = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{-h}{|-h|} = -1$$

and $f(0) = 0$

$\therefore$ **$f(x)$ is not continuous and hence not differentiable at $x = 0$.**

24. If $f(x) = x^5 \operatorname{sgn} x$, where $\operatorname{sgn} x = \begin{Bmatrix} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{Bmatrix}$, then $f(x)$ is

(A) differentiable as well as continuous at $x = 0$
(B) continuous but not differentiable at $x = 0$
(C) differentiable but not continuous at $x = 0$
(D) neither differentiable nor continuous at $x = 0$

Solution: (A)

Since,

$$\operatorname{sgn} x = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \text{ or } \operatorname{sgn} x = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$$\text{Therefore, } f(x) = x^5 \operatorname{sgn} x = \begin{cases} x^5, & x > 0 \\ 0, & x = 0 \\ -x^5, & x < 0 \end{cases}$$

Clearly, **$f(x)$ is continuous as well as differentiable at $x = 0$.**

25. Let $f(x) = |x|$ and $g(x) = [x]$, where [.] denotes the greatest integer function. Then (fog) '(– 2) is

(A) 0
(B) does not exist
(C) – 1
(D) 1

Solution: (B)

$$(fog)(x) = f(g(x)) = f([x]) = |[x]|$$

$$\text{Now, L}(fog)'(-2) = \lim_{h\to 0} \frac{(fog)(-2-h)-(fog)(-2)}{-h}$$

$$= \lim_{h\to 0} \frac{|[-2-h]| - |[-2]|}{-h}$$

$$= \lim_{h\to 0} \frac{|-3|-|-2|}{-h} = \lim_{h\to 0} \frac{-1}{h} \to -\infty$$

$\therefore$ **$(fog)'(-2)$ does not exist.**

26. Let $f(x + y) = f(x) f(y)$ for all x, y, where $f(0) \neq 0$. If $f'(0) = 2$, then $f(x)$ is equal to

(A) Ae^x
(B) Ae^{2x}
(C) $2x$
(D) None of these

Solution: (B)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x+0)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)\cdot f(h)-f(x)\cdot f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)-f(0)}{h}\cdot f(x)$$

$$= f'(0)\cdot f(x) = 2f(x). \quad (\because f'(0) = 2)$$

Now, $\frac{df}{dx} = 2f$ or $\frac{df}{f} = 2\,dx \Rightarrow d(\log f - 2x) = 0$

$\therefore \log f - 2x = c, \therefore f = e^{2x+c} = e^c \times e^{2x} = Ae^{2x}$,

where $A = e^c$ = constant.

27. Let $f(x + y) = f(x)\cdot f(y)$ for all x, y where $f(0) \neq 0$. If $f(5) = 2$ and $f'(0) = 3$, then $f'(5)$ is equal to

(A) 6
(B) 0
(C) 1
(D) None of these

Solution: (A)

$$f'(5) = \lim_{h\to 0} \frac{f(5+h)-f(5)}{h}$$

$$= \lim_{h\to 0} \frac{f(5+h)-f(5+0)}{h}$$

$$= \lim_{h\to 0} \frac{f(5)\cdot f(h)-f(5)\cdot f(0)}{h}$$

$[\because f(x + y) = f(x)\cdot f(y)$ for all $x, y]$

$$= \left(\lim_{h\to 0} \frac{f(h)-f(0)}{h}\right)\cdot f(5)$$

$$= f'(0)\cdot f(5) = 3\cdot 2 = \mathbf{6}$$

28. Let $f: R \to R$ be a function such that $f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}$, $f(0) = 0$ and $f'(0) = 3$. Then

(A) $f(x)$ is a quadratic function
(B) $f(x)$ is continuous but not differentiable
(C) $f(x)$ is differentiable in R
(D) $f(x)$ is bounded in R

Solution: (C)

We have,

$$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}, f(0) = 0 \text{ and } f'(0) = 3$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(\frac{3x+3h}{3}\right) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(3x)+f(3h)}{3} - \frac{f(3x)+f(0)}{3}}{h}$$

$$= \lim_{h\to 0} \frac{f(3h)-f(0)}{3h} = 3$$

$\therefore \quad f(x) = 3x + c, \because f(0) = 0 \Rightarrow c = 0$

$\therefore \quad f(x) = 3x$

29. If $f(x+y) = 2f(x)\cdot f(y)$ for all x, y, where $f'(0) = 3$ and $f(4) = 2$, then $f'(4)$ is equal to

(A) 6 (B) 12
(C) 4 (D) None of these

Solution: (B)

$$f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h}$$

$$= \lim_{h\to 0} \frac{f(4+h)-f(4+0)}{h}$$

$$= \lim_{h\to 0} \frac{2f(4)\cdot f(h) - 2f(4)f(0)}{h}$$

[Using $f(x+y) = 2f(x)\cdot f(y)$ for all x, y]

$$= 2f(4) \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

$$= 2f(4) \times f'(0) = 2\times 2\times 3 = 12$$

30. If a function $f: R \to R$ be such that $f(x+y) = f(x)\cdot f(y)$ for all $x, y \in R$ where $f(x) = 1 + x\,\phi(x)$ and $\lim_{x\to 0} \phi(x) = 1$, then

(A) $f'(x)$ does not exist
(B) $f'(x) = 2f(x)$ for all x
(C) $f'(x) = f(x)$ for all x
(D) None of these

Solution: (C)

$$Rf'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)\,f(h)-f(x)}{h}$$

$$= f(x) \lim_{h\to 0} \frac{1+h\,\phi(h)-1}{h}$$

$$= f(x) \lim_{h\to 0} \phi(h)$$

$$= f(x)\cdot 1 = f(x)$$

and $$Lf'(x) = \lim_{h\to 0} \frac{f(x-h)-f(x)}{-h}$$

$$= \lim_{h\to 0} \frac{f(x)\,f(-h)-f(x)}{-h}$$

$$= f(x) \lim_{h\to 0} \frac{f(-h)-1}{-h}$$

$$= f(x) \lim_{h\to 0} \frac{1-h\,\phi(-h)-1}{-h}$$

$$= f(x)\cdot 1 = f(x).$$

Hence **$f'(x)$ exists and is equal to $f(x)$.**

31. Let $f(x+y) = f(x)\cdot f(y)$ for all $x, y \in R$ and $f(x) = 1 + x\,\phi(x)\log 2$ where $\lim_{x\to 0} \phi(x) = 1$. Then $f'(x)$ is equal to

(A) $\log 2^{f(x)}$ (C) $\log [f(x)]^2$
(C) $\log 2$ (D) None of these

Solution: (A)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)\cdot f(h)-f(x)}{h}$$

[$\because f(x+y) = f(x)\cdot f(y)$]

$$= f(x) \lim_{h\to 0} \left(\frac{f(h)-1}{h}\right)$$

$$= f(x) \lim_{h\to 0} \frac{1+h\,\phi(h)\log 2 - 1}{h}$$

[$\because f(x) = 1 + x\,\phi(x)\log 2$]

$$= f(x)\log 2 \lim_{h\to 0} \phi(h)$$

$$= f(x)\times \log 2\times 1 \quad \left[\because \lim_{h\to 0}\phi(h) = 1\right]$$

$$= \log 2^{f(x)}$$

32. Let $f(x+y) = f(x)\cdot f(y)$ and $f(x) = 1 + x\,g(x)\,G(x)$ where $\lim_{x\to 0} g(x) = a$ and $\lim_{x\to 0} G(x) = b$. Then $f'(x) = k\,f(x)$, where k is equal to

(A) $\frac{a}{b}$ (B) $1 + ab$
(C) ab (D) None of these

Solution: (C)

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h}$$

$[\because f(x+y) = f(x)\cdot f(y)]$

$$= f(x)\cdot \lim_{h\to 0}\left(\frac{f(h)-1}{h}\right)$$

$$= f(x)\lim_{h\to 0}\frac{1+h\,g(h)\,G(h)-1}{h}$$

$$= f(x)\lim_{h\to 0} g(h)\,G(h)$$

$$= f(x)\lim_{h\to 0} g(h)\lim_{h\to 0} G(h)$$

$$= ab\,f(x).\ \therefore\ k = ab$$

33. Let f and g be differentiable functions satisfying $g'(a) = 2$, $g(a) = b$ and fog = I (identity function). Then $f'(b)$ is equal to

(A) 2 (B) $\frac{2}{3}$

(C) $\frac{1}{2}$ (D) None of these

Solution: (C)

We have, $fog = I$

$\Rightarrow \quad (fog)(x) = x$ for all x

$\Rightarrow \quad f[g(x)] = x \Rightarrow f'[g(x)]\times g'(x) = 1$

$\Rightarrow \quad f'[g(a)]\times g'(a) = 1$

$\Rightarrow \quad f'[g(a)] = \frac{1}{g'(a)} = \frac{1}{2}$ $[\because g'(a) = 2]$

$\Rightarrow \quad f'(b) = \frac{1}{2}$ $[\because g(a) = b]$

34. If $4x + 3|y| = 5y$, then y as a function of x is

(A) not continuous at $x = 0$
(B) not defined for all real x
(C) $\frac{dy}{dx} = \frac{1}{2}$ for $x < 0$
(D) derivable at $x = 0$

Solution: (C)

We have, $4x + 3|y| = 5y$

$\Rightarrow \quad 4x + 3y = 5y$ if $y \ge 0$

and $4x - 3y = 5y$ if $y < 0 \Rightarrow y = \begin{cases} 2x, & x \ge 0 \\ \frac{1}{2}x, & x < 0 \end{cases}$

Clearly, y is continuous at $x = 0$ but not differentiable at $x = 0$.

Also, $\frac{dy}{dx} = \begin{cases} 2, & x \ge 0 \\ \mathbf{\frac{1}{2}}, & \boldsymbol{x < 0} \end{cases}$

35. The function $f(x) = \max.\{(1-x), (1+x), 2\}, x \in (-\infty, \infty)$, is

(A) continuous at all points
(B) differentiable at all points
(C) differentiable at all points except at $x = 1$ and $x = -1$
(D) continuous at all points except at $x = 1$ and $x = -1$, where it is discontinuous

Solution: (A, C)

$$f(x) = \begin{cases} 1-x, & x \le -1 \\ 2, & -1 < x \le 1 \\ 1+x, & x > 1 \end{cases}$$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}(1-x) = 2 = \lim_{x\to -1^+} f(x)$$

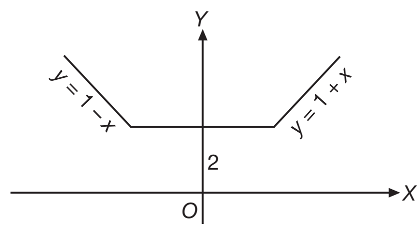
and $\lim_{x\to 1} f(x) = 2$, so f is continuous at all points.

$$f'(-1^-) = \lim_{h\to 0^-}\frac{f(-1+h)-f(-1)}{h}$$

$$= \lim_{h\to 0^-}\frac{1+1-h-2}{h} = -1$$

$$f'(-1^+) = 0$$

Similarly, $f'(1^-) = 0$ and $f'(1+) = 1$, so f is differentiable everywhere except at $x = -1, 1$.



36. Let $f: R \to R$ be a function defined by $f(x) = \max\{x, x^3\}$. The set of all points where $f(x)$ is not differentiable is

(A) $\{-1, 1\}$ (B) $\{-1, 0\}$
(C) $\{0, 1\}$ (D) $\{-1, 0, 1\}$

Solution: (D)

If $x < -1$, then $x > x^3$. So, $f(x) = x$.

If $x = -1$, then $x = x^3$. So, $f(x) = x$.

If $-1 < x < 0$, then $x < x^3$. So, $f(x) = x^3$.

If $x = 0$, then $x = x^3$. So, $f(x) = x^3$.

If $0 < x < 1$, then $x > x^3$. So, $f(x) = x$.

If $x = 1$, then $x = x^3$. So, $f(x) = x$.

If $x > 1$, then $x < x^3$. So, $f(x) = x^3$.

$$\text{Thus, } f(x) = \begin{array}{lll} x, & x & \le -1 \\ x^3, & -1 & < x \le 0 \\ x, & 0 & < x \le 1 \\ x^3, & x & > 1. \end{array}$$

Clearly, $f(x)$ is not differentiable at $x = -1, 0, 1$.

37. If $g(x) = (x^2 + 2x + 3) f(x)$, $f(0) = 5$ and $\lim\limits_{x \to 0} \dfrac{f(x) - 5}{x} = 4$, then $g'(0)$ is equal to

(A) 22 (B) 20

(C) 18 (D) None of these

Solution: (A)

We have, $\lim\limits_{x \to 0} \dfrac{f(x) - 5}{x} = 4$

$$\Rightarrow \quad \lim_{x \to 0} \frac{f(x) - f(0)}{x} = 4 \qquad [\because f(0) = 5]$$

$$\Rightarrow \quad f'(0) = 4.$$

Since, $g(x) = (x^2 + 2x + 3) f(x)$

$$\Rightarrow \quad g'(x) = (x^2 + 2x + 3) f'(x) + (2x + 2) f(x)$$

$$\therefore \quad g'(0) = 3 f'(0) + 2 f(0) = 3(4) + 2(5) = 22.$$

38. The points where the function $f(x) = [x] + |1 - x|$, $-1 \le x \le 3$, where [.] denotes the greatest integer function, is not differentiable, are

(A) $x = -1, 0, 1, 2, 3$

(B) $x = -1, 0, 2$

(C) $x = 0, 1, 2, 3$

(D) $x = -1, 0, 1, 2$

Solution: (C)

We have,

$$f(x) = [x] + |1 - x|, -1 \le x \le 3$$

$$= \begin{cases} -x & , -1 \le x < 0 \\ 1 - x & , 0 \le x < 1 \\ x & , 1 \le x < 2 \\ 1 + x & , 2 \le x < 3 \\ 5 & , x = 3 \end{cases}$$

The only doubtful points are $x = -1, 0, 1, 2$ and 3. It can be easily seen that $f(x)$ is differentiable at $x = -1$ but not differentiable at $x = 0, 1, 2$ and 3.
Hence, the required points are 0, 1, 2 and 3.

39. Let $f(x) = a + b|x| + c|x|^4$, where a, b and c are real constants. Then $f(x)$ is differentiable at $x = 0$ if

(A) $a = 0$ (B) $b = 0$

(C) $c = 0$ (D) None of these

Solution: (B)

Since $f(x)$ is differentiable at $x = 0$, therefore,

$$L f'(0) = R f'(0)$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{f(0 - h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{(a + b|-h| + c|-h|^4) - a}{-h}$$

$$= \lim_{h \to 0} \frac{(a + b|h| + c|h|^4) - a}{h}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{bh + ch^4}{-h} = \lim_{h \to 0} \frac{bh + ch^4}{h}$$

$$\Rightarrow \quad \lim_{h \to 0} (-b - ch^3) = \lim_{h \to 0} (b + ch^3)$$

$$\Rightarrow \quad -b = b \text{ i.e. } b = 0$$

40. If $f(x) = \dfrac{1}{[\sin x]}$, where [.] denotes the greatest integer function, then

(A) Domain of $f(x)$ is $(2n\pi + \pi, 2n\pi + 2\pi) \cup \left\{2n\pi + \dfrac{\pi}{2}\right\}$ where $n \in I$

(B) $f(x)$ is continuous when $x \in (2n\pi + \pi, 2n\pi + 2\pi)$

(C) $f(x)$ is not differentiable at $x = \pi/2$

(D) None of these

Solution: (A, C)

The function $f(x)$ is defined when $-1 \le \sin x < 0$ or $\sin x = 1$.

$\Rightarrow x \in ((2n+1)\pi, (2n+2)\pi) \cup \left\{2n\pi + \frac{\pi}{2}\right\}, n \in I.$

When $x \in (2n\pi + \pi, 2n\pi + 2\pi), f(x) = -1$

$\therefore f(x)$ is a constant function.

Hence $f(x)$ is continuous when $x \in (2n\pi + \pi, 2n\pi + 2\pi)$.

Now, $Lf'(\pi/2) = \lim_{h \to 0} \dfrac{f\left(\frac{\pi}{2} - h\right) - f\left(\frac{\pi}{2}\right)}{-h}$

$$= \lim_{h \to 0} \frac{\dfrac{1}{\left[\sin\left(\frac{\pi}{2} - h\right)\right]} - 1}{-h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1}{[\cos h]} - 1}{-h}$$

= does not exist.

Hence, $f(x)$ is not differentiable at $x = \pi/2$.

EXERCISES

Single Option Correct Type

1. If $f(x) = \begin{cases} 3, & x < 0 \\ 2x + 1, & x \ge 0 \end{cases}$, then

(A) both $f(x)$ and $f(|x|)$ are differentiable at $x = 0$
(B) $f(x)$ is differentiable but $f(|x|)$ is not differentiable at $x = 0$
(C) $f(|x|)$ is differentiable but $f(x)$ is not differentiable at $x = 0$
(D) both $f(x)$ and $f(|x|)$ are not differentiable at $x = 0$

2. Let $f(x) = \cos x$ and $g(x) = [x + 2]$, where [.] denotes the greatest integer function. Then, $(gof)'\left(\frac{\pi}{2}\right)$ is

(A) 1 (B) 0
(C) -1 (D) does not exist

3. Let $f(x) = \begin{cases} \dfrac{1}{|x|} & |x| \ge 1 \\ ax^2 + b & |x| < 1 \end{cases}$. If $f(x)$ is continuous and differentiable at any point, then

(A) $a = \frac{1}{2}, b = -\frac{3}{2}$ (B) $a = -\frac{1}{2}, b = \frac{3}{2}$
(C) $a = 1, b = -1$ (D) None of these

4. Let $f(x)$ be a function satisfying $f(x + y) = f(x) f(y)$ for all $x, y \in R$. If $f(x) = 1 + x\phi(x) + x^2\phi(x)\psi(x)$, where $\lim_{x \to 0} \phi(x) = a$ and $\lim_{x \to 0} \psi(x) = b$, then $f'(x)$ is equal to

(A) $(a + b) f(x)$ (B) $af(x)$
(B) $bf(x)$ (D) None of these

5. The function $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi$, where [.] denotes the greatest integer function, is discontinuous at

(A) all x (B) all integer points
(C) no x (D) x which is not an integer

6. The left-hand derivative of $f(x) = [x] \sin(\pi x)$ at $x = k$, k an integer and $[x]$ = greatest integer $\le x$, is

(A) $(-1)^k (k-1)\pi$
(B) $(-1)^{k-1} \cdot (k-1)\pi$
(C) $(-1)^k \cdot k\pi$
(D) $(-1)^{k-1} \cdot k\pi$.

7. If $\lim_{x \to a+} f(x) = l = \lim_{x \to a-} g(x)$ and $\lim_{x \to a-} f(x) = m = g(x)$, then the function $f(x) \cdot g(x)$

(A) is not continuous at $x = a$
(B) has a limit when $x \to a$ and it is equal to lm
(C) is continuous at $x = a$
(D) has a limit when $x \to a$ but it is not equal to lm

8. Let $[x]$ denotes the greatest integer less than or equal to x. If $f(x) = [x \sin \pi x]$, then $f(x)$ is

(A) continuous at $x = 0$
(B) continuous in $(-1, 0)$
(C) differentiable at $x = 1$
(D) differentiable in $(-1, 1)$

9. The function $f(x) = [x]^2 - [x^2]$ (where $[x]$ is the greatest integer less than or equal to x), is discontinuous at
(A) all integers
(B) all integers except 0 and 1
(C) all integers except 0
(D) all integers except 1

10. Let $f: R \to R$ be any function. Define $g: R \to R$ by $g(x) = |f(x)|$ for all x. Then g is
(A) onto if f is onto
(B) one-one if f is one-one
(C) continuous if f is continuous
(D) differentiable if f is differentiable

11. Let $f(x)$ be a function satisfying the condition $f(-x) = f(x)$, for all real x. If $f'(0)$ exists, then its value is
(A) 0 (B) 1
(C) – 1 (D) None of these

12. If $f(x) = \begin{cases} \dfrac{x(3e^{1/x}+4)}{2-e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is
(A) continuous as well as differentiable at $x = 0$
(B) continuous but not differentiable at $x = 0$
(C) differentiable but not continuous at $x = 0$
(D) None of these

13. The function $f(x) = \dfrac{1}{u^2 + u - 2}$, where $u = \dfrac{1}{x-1}$, is discontinuous at the points
(A) $x = -2, 1, \frac{1}{2}$ (B) $x = \frac{1}{2}, 1, 2$
(C) $x = 1, 0$ (D) None of these

14. Let $f(x) = [3 + 2\cos x]$, $x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$, where [.] denotes the greatest integer function. The number of points of discontinuity of $f(x)$ is
(A) 3 (B) 2
(C) 5 (D) None of these

15. The set of points of continuity of the function

$f(x) = \sqrt{\dfrac{1}{2} - \cos^2 x}$ is

(A) $\left\{x : \dfrac{\pi}{4} + 2n\pi \le x \le \dfrac{3\pi}{4} + 2n\pi, n \in I\right\}$

(B) $\left\{x : \dfrac{5\pi}{4} + 2n\pi \le x \le \dfrac{7\pi}{4} + 2n\pi, n \in I\right\}$

(C) $\left\{x : \dfrac{\pi}{4} + 2n\pi \le x \le \dfrac{3\pi}{4} + 2n\pi\right\} \cup \left\{x : \dfrac{5\pi}{4} + 2n\pi \le x \le \dfrac{7\pi}{4} + 2n\pi\right\}$

(D) None of these

16. If $f(x) = \dfrac{1}{1-x}$, then the points of discontinuity of the function $f^{3n}(x)$, where $f^n = fof \ldots of$ (n times), are
(A) $x = 2$ (B) $x = 0$
(C) $x = 1$ (D) continuous everywhere

17. The function $f(x) =$ are tan $\dfrac{1}{x-5}$ has
(A) discontinuity of the first kind at $x = 5$
(B) discontinuity of the second kind at $x = 5$
(C) removable discontinuity at $x = 5$
(D) continuous at $x = 5$

18. If $f(x) = \sum_{k=0}^{n} a_k |x-1|^k$, where $a_i \in R$ then
(A) $f(x)$ is continuous at $x = 1$ for all $a_k \in R$
(B) $f(x)$ is differentiable at $x = 1$ for all $a_k \in R$
(C) $f(x)$ is differentiable at $x = 1$, provided $a_{2k+1} = 0$
(D) $f(x)$ is continuous at $x = 1$, provided $a_{2k} = 0$

19. If $f(x) = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right)$, then $f(x)$ is differentiable on
(A) $(-\infty, \infty)$ (B) $(-\infty, \infty) \setminus \{0\}$
(C) $(-\infty, \infty) \setminus \{-1, 1\}$ (D) None of these

20. The set of points of discontinuities of the function $f(x) = \sqrt{x} - [\sqrt{x}]$, where [x] denotes the greatest integer less than or equal to x, contains the set
(A) $(-\infty, 0)$ (B) $\{n^2 : n \in N\}$
(C) N (D) $\{2n : n \in N\}$

21. If $f(x) = |3 - x| + (3 + x)$, where (x) denotes the least integer greater than or equal to x, then
(A) $f(x)$ is continuous as well as differentiable at $x = 3$
(B) $f(x)$ is continuous but not differentiable at $x = 3$
(C) $f(x)$ is differentiable but not continuous at $x = 3$
(D) $f(x)$ is neither differentiable nor continuous at $x = 3$

22. Let

$$f(x) = \begin{cases} \dfrac{1+\cos x}{(\pi - x)^2} \cdot \dfrac{\sin^2 x}{\log(1 + \pi^2 - 2\pi x + x^2)}, & x \neq \pi \\ k, & x = \pi \end{cases}.$$

If $f(x)$ is continuous at $x = \pi$, then k is equal to

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$

(C) $\frac{-1}{2}$ (D) $\frac{-1}{4}$

23. Let $f(x)$ be a continuous function defined for $1 \le x \le 3$. If $f(x)$ takes rational values for all x and $f(2) = 10$, then $f(1\cdot5)$ is equal to

(A) 0 (B) 10

(C) not defined (D) any constant

24. If $f(x) = \int_0^x t \cos \frac{1}{t}\, dt$, then the number of points of discontinuity of $f(x)$ in the interval $(0, \pi)$ is

(A) 1 (B) 2

(C) 0 (D) None of these

25. If $f(x) = (-1)^{[x^3]}$, where [.] denotes the greatest integer function, then

(A) $f(x)$ is discontinuous for $x = n^{1/3}$, where $n \in I$

(B) $f(3/2) = 1$

(C) $f'(x) = 0$ for $-1 < x < 1$

(D) None of these

26. If $f(x) = \left[\frac{1}{\sqrt{2}}(\cos x + \sin x)\right]$, $0 < x < 2\pi$, where [.] denotes the greatest integer function, then the number of points of discontinuity of $f(x)$ is

(A) 5 (B) 4

(C) 3 (D) None of these

27. Let $f(x) = a\,[x] + b\,e^{|x|} + c\,|x|^2$, where a, b and c are real constants. If $f(x)$ is differentiable at $x = 0$, then

(A) $b = 0, c = 0, a \in R$

(B) $a = 0, c = 0, b \in R$

(C) $a = 0, b = 0, c \in R$

(D) None of these

28. If $f(x) = [x] \sin\left(\frac{\pi}{[x+1]}\right)$, where [.] denotes the greatest integer function, then the points of discontinuity of f in the domain are

(A) Z (B) $Z \backslash \{0\}$

(C) $R \backslash [-1, 0)$ (D) None of these

29. Let f be a function satisfying $f(x + y) = f(x) + f(y)$ and $f(x) = x^3 \phi(x)$ for all x and y, where $\phi(x)$ is a continuous function then $f'(x)$ is equal to

(A) $g(0)$ (B) $g'(x)$

(C) 0 (D) None of these

30. The value of $f(0)$ so that the function

$f(x) = \frac{\sqrt[3]{1+x} - \sqrt[4]{1+x}}{x}$ becomes continuous at $x = 0$, is

(A) $\frac{1}{12}$ (B) $\frac{7}{12}$

(C) 0 (D) None of these

31. If f is an even function such that $\lim_{h \to 0} \frac{f(h) - f(0)}{h}$ has some finite non-zero value, then

(A) f is continuous and derivable at $x = 0$

(B) f is continuous but not derivable at $x = 0$

(C) f may be discontinuous at $x = 0$

(D) None of these

32. If f is differentiable function satisfying $f(0) = 0$, and if $g(x) = \frac{f(x)}{x}$, then the value, that should be assigned to $g(0)$, so that g is continuous at '0' is

(A) 1 (B) 0

(C) $f(0)$ (D) $f'(0)$

33. Let $f(x) = \frac{1}{[\sin x]}$, [.] being the greatest integer function, then

(A) $f(x)$ is not continuous, where $x \in (2n\pi, 2n\pi + \pi)$, $n \in I$

(B) $f(x)$ is differentiable at $x = \frac{\pi}{4}$

(C) $f(x)$ is differentiable at $x = \frac{\pi}{2}$

(D) None of these

34. The function

$$f(x) = \begin{cases} 1 - 2x + 3x^2 - 4x^3 + \ldots \text{ to } \infty, & x \ne -1 \\ 1, & x = -1 \end{cases} \text{ is}$$

(A) continuous and derivable at $x = -1$

(B) neither continuous nor derivable at $x = -1$

(C) continuous but not derivable at $x = -1$

(D) None of these

35. If $f(x) = x + \frac{x}{1+x} + \frac{x}{(1+x)^2} + \ldots$ to ∞, then at $x = 0$, $f(x)$

(A) has no limit

(B) is discontinuous

(C) is continuous but not differentiable

(D) is differentiable

36. The function $f(x) = [x^2] + [-x]^2$, where [.] denotes the greatest integer function, is

(A) continuous and derivable at $x = 2$
(B) neither continuous nor derivable at $x = 2$
(C) continuous but not derivable at $x = 2$
(D) None of these

37. If the function

$$f(x) = \begin{cases} (1-|\tan x|)^{\frac{a}{|\tan x|}} & , \frac{-\pi}{4} < x < 0 \\ b & , x = 0 \\ e^{\frac{\sin 3x}{\sin 2x}} & , 0 < x < \frac{\pi}{4} \end{cases}$$

is continuous at $x = 0$, then

(A) $a = \frac{-3}{2}\ b = \frac{3}{2}$ (B) $a = \frac{3}{2}\ b = e^{3/2}$

(C) $a = \frac{-3}{2},\ b = e^{3/2}$ (D) None of these

38. If $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} (\log a)^n$, then at $x = 0$, $f(x)$
(A) has no limit
(B) is discontinuous
(C) is continuous but not differentiable
(D) is differentiable

39. The values of constants a and b so as to make the function $f(x) = \begin{cases} \frac{1}{|x|} & , |x| \ge 1 \\ ax^2 + b, & |x| < 1 \end{cases}$ continuous as well as differentiable for all x, are

(A) $a = \frac{-1}{2},\ b = \frac{3}{2}$ (B) $a = \frac{1}{2},\ b = \frac{3}{2}$

(C) $a = \frac{-1}{2},\ b = \frac{-3}{2}$ (D) None of these

40. If $f(x) = [\tan^2 x]$ (where [.] denotes the greatest integer function), then
(A) $\lim_{x \to 0} f(x)$ does not exist
(B) $f(x)$ is continuous at $x = 0$
(C) $f(x)$ is non-differentiable at $x = 0$
(D) $f(0) = 1$.

41. The values of p and q for which the function

$$f(x) = \begin{cases} \frac{\sin(p+1)x + \sin x}{x}, & x < 0 \\ q, & x = 0 \\ \frac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}}, & x = 0 \end{cases}$$

is continuous for all x in R, are

(A) $p = \frac{1}{2},\ q = \frac{3}{2}$ (B) $p = \frac{1}{2},\ q = -\frac{3}{2}$

(C) $p = \frac{5}{2},\ q = \frac{1}{2}$ (D) $p = -\frac{3}{2},\ q = \frac{1}{2}$

42. If $f : R \to R$ is a fucntion defined by $f(x) = [x] \cos \left(\frac{2x-1}{2}\right)\pi$, where $[x]$ denotes the greatest integer fucntion, then f is
(A) continuous for every real x
(B) discontinuous only at $x = 0$
(C) discontinuous only at non-zero integral values of x
(D) continuous only at $x = 0$

43. Consider the fucntion, $f(x) = |x - 2| + |x - 5|, x \in R$.
Statement-1: $f'(4) = 0$
Statement-2: f is continuous in [2, 5] differentiable in (2, 5) and $f(2) = f(5)$
(A) Statement-1 is false, Statement-2 is true
(B) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1
(C) Statement-1 is true, statement-2 is true; statement-2 is **not** a correct explanation for statement-1
(D) Statement-1 is true, statement-2 is false

44. If $f(x) = |x| + [x - 1]$, where [.] is greatest integer function, then $f(x)$ is:
(A) continuous at $x = 0$ as well as at $x = 1$
(B) continous at $x = 0$ but not at $x = 1$
(C) continuous at $x = 1$ but not at $x = 0$
(D) neither continuous at $x = 0$ nor at $x = 1$

45. Amongst the following functions, a function that is differentiable at $x = 0$ is
(A) $\cos(|x|) - |x|$
(B) $\cos(|x|) + |x|$
(C) $\sin(|x|) + |x|$
(D) $\sin(|x|) - |x|$

46. Let $f(x) = x^2 - 8x + 12, x \in [2, 6]$.
Statement-1: $f'(c) = 0$ for some $c \in (2, 6)$
Statement-2: f is continuous on [2, 6] and differentiable on (2, 6) with $f(2) = f(6)$
(A) Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanantion for Statement-1
(B) Statement-1 is true, Statement-2 is true, Statement-2 is **not** a correct expalantion for Statement-1
(C) Statement-1 is true, Statement-2 is false
(D) Statement-1 is false, Statement-2 is true

47. Let $f(x) = \begin{cases} (x-1)\sin\left(\frac{1}{x-1}\right) & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$. Then which of the following is true?

(A) f is neither differentiable at $x = 0$ nor at $x = 1$
(B) f is differentiable at $x = 0$ and at $x = 1$
(C) f is differentiable at $x = 0$ but not at $x = 1$
(D) f is differentiable at $x = 1$ but not at $x = 0$

48. Let $f(x) = x\,|x|$ and $g(x) = \sin x$.

Statement-1: gof is differentiable at $x = 0$ and its derivative is continuous at that point.

Statement-2: gof is twice differentiable at $x = 0$.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1
(C) Statement-1 is true, Statement-2 is false
(D) Statement-1 is false, Statement-2 is true

49. The function $f(x) = \begin{cases} (x+1)^{2-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is

(A) discontinuous at only one point
(B) discontinuous exactly at two points
(C) continuous everywhere
(D) None of these

50. $f(x) = \begin{cases} \dfrac{e^{[x]+|x|}-2}{[x]+|x|}, & x \neq 0 \\ -1, & x = 0 \end{cases}$, ([.] denotes the greatest integer function), then

(A) $f(x)$ is continuous at $x = 0$
(B) $\lim_{x \to 0^+} f(x) = -1$
(C) $\lim_{x \to 0^-} f(x) = 1$
(D) None of these

51. The Dirichlet function, defined as

$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$, is

(A) continuous for all real x
(B) continuous only at some values of x
(C) discontinuous for all real x
(D) discontinuous only at some values of x

52. Let $f : R \to R$ be a function such that

$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}$, $f(0) = 0$ and $f'(0) = 3$.

Then,

(A) $f(x)$ is a quadratic function
(B) $f(x)$ is continuous but not differentiable
(C) $f(x)$ is differentiable in R
(D) $f(x)$ is bounded in R

53. If $f(x) = \begin{cases} x, \text{ when } x \text{ is rational} \\ 1-x, \text{ when } x \text{ is irrational} \end{cases}$, then

(A) $f(x)$ is continuous for all real x
(B) $f(x)$ is discontinuous for all real x
(C) $f(x)$ is continuous only at $x = 1/2$
(D) $f(x)$ is discontinuous only at $x = 1/2$.

54. The points where the function $f(x) = [x] + |1 - x|$, $-1 \le x \le 3$, where [.] denotes the greatest integer function, is not differentiable, are

(A) $x = -1, 0, 1, 2, 3$
(B) $x = -1, 0, 2$
(C) $x = 0, 1, 2, 3$
(D) $x = -1, 0, 1, 2$

55. Let a function $f : R \to R$ satisfy the equation $f(x + y) = f(x) + f(y)$ for all x, y. If the function $f(x)$ is continuous at $x = 0$, then

(A) $f(x) = 0$ continuous for all x
(B) $f(x)$ is continuous for all positive real x
(C) $f(x)$ is continuous for all x
(D) None of these

56. The function $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi$, where [.] denotes the greatest integer function, is discontinuous at

(A) all x
(B) all integer points
(C) no x
(D) x which is not an integer.

57. The function $f(x) = [x]^2 - [x^2]$ (where $[x]$ is the greatest integer less than or equal to x), is discontinuous at

(A) all integers
(B) all integers except 0 and 1
(C) all integers except 0
(D) all integers except 1

58. The function $f(x) = \frac{1}{u^2+u-2}$, where $u = \frac{1}{x-1}$, is discontinuous at the points

(A) $x = -2, 1, \frac{1}{2}$
(B) $x = \frac{1}{2}, 1, 2$
(C) $x = 1, 0$
(D) None of these

59. If $f(x) = \sum_{k=0}^{n} a_k |x-1|^k$, where $a_i \in R$, then

(A) $f(x)$ is continuous at $x = 1$ for all $a_k \in R$
(B) $f(x)$ is differentiable at $x = 1$ for all $a_k \in$ R
(C) $f(x)$ is differentiable at $x = 1$, provided $a_{2k+1} = 0$
(D) $f(x)$ is continuous at $x = 1$, provided $a_{2k} = 0$

60. If $f(x) = [x] \sin\left(\dfrac{\pi}{[x+1]}\right)$, where [.] denotes the greatest integer function, then the points of discontinuity of f in the domain are

(A) Z (B) $Z\backslash\{0\}$
(C) $R\backslash[-1, 0)$ (D) None of these

61. If f is differentiable function satisfying $f(0) = 0$ and if $g(x) = \dfrac{f(x)}{x}$, then value, that should be assigned to $g(0)$, so that g is continuous at '0' is

(A) 1 (B) 0
(C) $f(0)$ (D) $f'(0)$

62. The value of $f(0)$ so that the function $f(x) = \dfrac{\cos^{-1}(1-\{x\}^2)\sin^{-1}(1-\{x\})}{\{x\}-\{x\}^3}$, $x \neq 0$ ($\{x\}$ denotes fractional part of x) becomes continuous at $x = 0$ is

(A) $\dfrac{\pi}{\sqrt{2}}$ (B) $\dfrac{\pi}{4}$
(C) $\dfrac{\pi}{2}$ (D) None of these

63. If the function $f(x)$ defined as

$$f(x) = \begin{cases} (\sin x + \cos x)^{\operatorname{cosec} x}, & \dfrac{-\pi}{2} < x < 0 \\ a, & x = 0 \\ \dfrac{e^{1/x} + e^{2/x} + e^{3/x}}{ae^{-2+1/x} + be^{-1+3/x}}, & 0 < x < \dfrac{\pi}{2} \end{cases}$$

is continuous at $x = 0$, then

(A) $a = e, b = 1$ (B) $a = 1, b = e$
(C) $a = \dfrac{1}{e}, b = 1$ (D) None of these

64. If the function $f(x) = \dfrac{(128a + ax)^{1/8} - 2}{(32 + bx)^{1/5} - 2}$ is continuous at $x = 0$, then the value of $\dfrac{a}{b}$ is

(A) $\dfrac{3}{5}f(0)$ (B) $2^{8/5}f(0)$
(C) $\dfrac{64}{5}f(0)$ (D) None of these

65. Let $f: R^+ \to R$ satisfies the equation $f(xy) = e^{xy-x-y}\{e^y f(x) + e^x f(y)\}, \forall x, y \in R^+$. If $f'(1) = e$, then $f(x) =$

(A) $e^x \log|x|$ (B) $e^{-x}\log|x|$
(C) $e^{2x}\log|x|$ (D) None of these

66. Let $f(x) = [n + p \sin x]$, $x \in (0, \pi)$, $n \in Z$ and p is a prime number, where $[\cdot]$ denotes the greatest integer function. Then, the number of points where $f(x)$ is not differentiable, are

(A) 0 (B) $2(p-1)$
(C) $2p - 1$ (D) None of these

67. The function $y = f(x)$, defined parametrically as $x = 2t - |t-1|$ and $y = 2t^2 + t|t|$, is

(A) continuous and differentiable for $x \in R$
(B) continuous for $x \in R$ and differentiable for $x \in R - \{2\}$.
(C) continuous for $x \in R$ and differentiable for $x \in R - \{-1, 2\}$
(D) None of these

68. If $f(x)$ is a continuous function for all real values of x satisfying $x^2 + (f(x) - 2)x + 2\sqrt{3} - 3 - \sqrt{3}f(x) = 0$, $\forall x \in R$, then the value of $f(\sqrt{3})$ is

(A) $\sqrt{3}$ (B) $1 - \sqrt{3}$
(C) $2(1 - \sqrt{3})$ (D) $2(\sqrt{3} - 1)$

69. The jump of the function at the point of discontinuity i.e., $x = 1$ of the function

$f(x) = \lim\limits_{n\to\infty} \dfrac{\log(2+x) - x^{2n}\sin x}{1 + x^{2n}}$ is

(A) $\sin 1 - \log 3$ (B) $\sin 1 + \log 3$
(C) $-\sin 1 + \log 3$ (D) None of these

70. The function $f(x) = \begin{cases} \dfrac{x-1}{e^{\frac{1}{x-1}} + 1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$

(A) is continuous
(B) has removable discontinuity
(C) has jump discontinuity
(D) has infinite discontinuity

71. Let $f: R \to R$ be a real valued function such that $|f(x) - f(y)| \leq |x - y|^2$ $\forall x, y \in R$. Then, the function $h(x) = \int f(x)\,dx$ is

(A) continuous $\forall x \in R$
(B) discontinuous at $x = 0$ only
(C) discontinuous at all integral points
(D) $h(0) = 0$

72. If f is a continuous function from R to R and $f(f(a)) = a$ for some $a \in R$, then the equation $f(x) = x$ has

(A) no solution
(B) exactly one solution
(C) at most one solution
(D) at least one solution

73. Let f be a continuous function on R such that $f(1/4n) = \left(\sin e^n\right)e^{-n^2} + \frac{n^2}{n^2+1}$. Then, the value of $f(0)$ is

(A) 1 (B) $\frac{1}{2}$

(C) 0 (D) None of these

74. Let f be a continuous and differentiable function in (a, b), $\lim_{x\to a^+} f(x) \to \infty$ and $\lim_{x\to b^-} f(x) \to -\infty$. If $f'(x) + f^2(x) \geq -1$ for $a < x < b$, then

(A) $b - a \leq \pi$ (B) $b - a\ ^3\pi$

(C) $b - a = \pi$ (D) None of these

75. Let f be a differentiable function such that $f(x + y) = f(x) + f(y) + 2xy - 1$ for all real x and y. If $f'(0) = \cos \alpha$, then $\forall x \in R$

(A) $f(x) = 0$ (B) $f(x) < 0$

(C) $f(x) > 0$ (D) $f(x) = x$

76. A function $f : R \to R$, where R is the set of real numbers satisfies the equation

$$f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y) + f(0)}{3}$$

for all x, y in R. If the function f is differentiable at $x = 0$, then f is

(A) linear (B) quadratic

(C) cubic (D) biquadratic

More than One Option Correct type

77. If $f(x) = \begin{cases} x^p \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then at $x = 0$, $f(x)$ is

(A) continuous if $p > 0$

(B) differentiable if $p > 1$

(C) continuous if $p > 1$

(D) differentiable if $p > 0$

78. Let $g(x) = x\ f(x)$, where $f(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

At $x = 0$,

(A) g is differentiable but g' is not continuous

(B) g is differentiable while f is not

(C) both f and g are differentiable

(D) g is differentiable and g' is continuous

79. The function $f(x) = \max. \{(1 - x), (1 + x), 2\}$, $x \in (-\infty, \infty)$, is

(A) continuous at all points

(B) differentiable at all points

(C) differentiable at all points except at $x = 1$ and $x = -1$.

(D) continuous at all points except at $x = 1$ and $x = -1$, where it is discontinuous.

80. The function $f(x) = (x)$, where (x) denotes the smallest integer $\geq x$, is

(A) continuous at integral points

(B) continuous at non-integral points

(C) discontinuous at integral points

(D) discontinuous at non-integral points

81. Let $f(x) = \begin{cases} \frac{1}{|x|} & |x| \geq 1 \\ ax^2 + b & |x| < 1 \end{cases}$. If f is continuous and differentiable at every point, then

(A) $a = \frac{1}{2}$ (B) $a = -\frac{1}{2}$

(C) $b = \frac{3}{2}$ (D) $b = \frac{-3}{2}$

82. Let $[x]$ denotes the greatest integer less than or equal to x. If $f(x) = [x \sin px]$, then $f(x)$ is

(A) continuous at $x = 0$

(B) continuous in $(-1, 0)$

(C) differentiable at $x = 1$

(D) differentiable in $(-1, 1)$

83. If $f(x) = \begin{cases} \frac{x\,(3e^{1/x} + 4)}{2 - e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is

(A) continuous at $x = 0$

(B) not continuous at $x = 0$

(C) differentiable at $x = 0$

(D) not differentiable at $x = 0$

84. If $f(x) = \frac{1}{1-x}$, then the points of discontinuity of the function $f^{3n}(x)$, where $f^n = fof \ldots$ of (n times), are

(A) $x = 2$ (B) $x = 0$

(C) $x = 1$ (D) continuous everywhere

85. The function

$$f(x)=\begin{cases}1-2x+3x^2-4x^3+\ldots \text{ to } \infty & , x\neq -1\\ 1 & , x=-1\end{cases}$$ is

(A) continuous at $x = -1$
(B) neither continuous nor derivable at $x = -1$
(C) derivable at $x = -1$
(D) not derivable at $x = -1$

86. The function $F(x)$, defined as

$$F(x)=\lim_{n\to\infty}\frac{f(x)+x^{2n}g(x)}{1+x^{2n}}$$

shall be continuous everywhere, if
(A) $f(1) = g(1)$ (B) $f(-1) = g(-1)$
(C) $f(1) = -g(1)$ (D) $f(-1) = -g(-1)$

87. If the function $f(x)$ defined as

$$f(x)=\begin{cases}3 & , x=0\\ \left(1+\dfrac{ax+bx^3}{x^2}\right)^{1/x} & , x>0\end{cases}$$

is continuous at $x = 0$, then
(A) $a = 0$ (B) $b = e^3$
(C) $a = 1$ (D) $b = ln\, 3$

88. If the function $f(x)=\dfrac{\sin 3x+a\sin 2x+b\sin x}{x^5}, x\neq 0$ is continuous at $x = 0$, then
(A) $a = -4$ (B) $b = 5$
(C) $a = 4$ (D) $f(0) = 1$

89. Let $f''(x)$ be continuous at $x = 0$.

If $\lim\limits_{x\to 0}\dfrac{2f(x)-3a\,f(2x)+bf(8x)}{\sin^2 x}$ exists and $f(0)\neq 0$, $f'(0)\neq 0$, then
(A) $a=\dfrac{-7}{9}$ (B) $b=\dfrac{1}{3}$
(C) $a=\dfrac{7}{9}$ (D) $b=-\dfrac{1}{3}$

90. If $f(x)=\begin{cases}x-3 & , x<0\\ x^2-3x+2, & x\geq 0\end{cases}$

and $g(x) = f(|x|) + |f(x)|$, then $g(x)$ is
(A) continuous in $R - \{0\}$
(B) continuous in R
(C) differentiable in $R - \{0, 1, 2\}$
(D) differentiable in $R - \{1, 2\}$

91. Let $f(x) = x^3 - x^2 + x + 1$ and

$$g(x)=\begin{cases}\max.\, f(t) & 0\leq t\leq x \text{ for } 0\leq x\leq 1\\ 3-x & 1<x\leq 2\end{cases}$$. Then, in the interval $[0, 2]$, $g(x)$ is
(A) continuous for all x
(B) differentiable for all x
(C) discontinuous at $x = 1$
(D) not differentiable at $x = 1$

92. Let $f(x) = x^4 - 8x^3 + 22x^2 - 24x$ and

$$g(x)=\begin{cases}\min.\, f(t) & x\leq t\leq x+1, -1\leq x\leq 1\\ x-10 & x>1\end{cases}$$

Then, in the interval $[-1, \infty)$, $g(x)$ is
(A) continuous for all x
(B) discontinuous at $x = 1$
(C) differentiable for all x
(D) not differentiable at $x = 1$

93. If $f(x) = [\tan x] + \sqrt{\tan x-[\tan x]},\ 0\leq x<\dfrac{\pi}{2}$, where $[\cdot]$ denotes the greatest integer function, then
(A) $f(x)$ is continuous in $\left[0,\dfrac{\pi}{2}\right)$
(B) $f(x)$ is not continuous at $x = 0$
(C) $f(x)$ is continuous at $x = 0, \dfrac{\pi}{4}$
(D) $f(x)$ has infinite points of discontinuity

94. Let $f(x)=g'(x)\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$, where g' is the derivative of g and is a continuous function, then $\lim\limits_{x\to 0}f(x)$ exists if
(A) $g(x)$ is a polynomial
(B) $g(x) = x$
(C) $g(x) = x^2$
(D) $g(x) = x^3h(x)$, where $h(x)$ is a polynomial

95. If the function $f(x)$, defined as

$$f(x)=\begin{cases}\dfrac{a(1-x\sin x)+b\cos x+5}{x^2} & , x<0\\ 3 & , x=0\\ \left[\left\{1+\left(\dfrac{cx+dx^3}{x^2}\right)\right\}\right]^{1/x} & , x>0\end{cases}$$

is continuous at $x = 0$, then
(A) $a = -1$ (B) $b = -4$
(C) $c = 0$ (D) $\log_e^3$

Passage Based Questions

Passage 1

Let f be a real valued function defined on an open internal $I \subset R$. If $x_0 \in I$, then we define g with domain $I - \{x_0\}$ by setting $g(x) = \dfrac{f(x) - f(x_0)}{x - x_0}$, for all $x \in I - \{x_0\}$. If $\lim\limits_{x \to x_0} g(x)$ exists and is finite, we denote it by $f'(x_0)$ and say that f is derivable at x_0. $f'(x_0)$ is called the derivative of f at x_0. If $\lim\limits_{x \to x_0^+} g(x)$ exists and is finite, we denote it by $Rf'(x_0)$ and say that f is derivable from right at x_0.

If $\lim\limits_{x \to x_0^-} g(x)$ exists and is finite, we denote it by $Lf'(x_0)$ and say that f is derivable from left at x_0.

It is obvious that f is derivable at x_0 iff $Lf'(x_0)$ and $Rf'(x_0)$ both exist and are equal. Also, if this condition be satisfied, then the common value is nothing else but $f'(x_0)$.

96. If $f(x) = \begin{cases} 3, x < 0 \\ 2x + 1, x \geq 0 \end{cases}$, then

(A) both $f(x)$ and $f(|x|)$ are differentiable at $x = 0$
(B) $f(x)$ is differentiable but $f(|x|)$ is not differentiable at $x = 0$
(C) $f(|x|)$ is differentiable but $f(x)$ is not differentiable at $x = 0$
(D) both $f(x)$ and $f(|x|)$ are not differentiable at $x = 0$.

97. Let $f(x) = \cos x$ and $g(x) = [x + 2]$, where [.] denotes the greatest integer function. Then, $(gof)'\left(\dfrac{\pi}{2}\right)$ is

(A) 1
(B) 0
(C) – 1
(D) does not exist

98. The left-hand derivative of $f(x) = [x] \sin(\pi x)$ at $x = k$, k an integer and $[x]$ = greatest integer $\leq x$, is

(A) $(-1)^k (k-1)\pi$
(B) $(-1)^{k-1} \cdot (k-1)\pi$
(C) $(-1)^k \cdot k\pi$
(D) $(-1)^{k-1} \cdot k\pi$.

Passage 2

Let f be a real-valued function defined on an interval I. If f be derivable at a point $x_0 \in I$, then it is continuous at x_0. The converse of the above statement does not hold. That is, a function may be continuous at a point but may fail to be derivable at that point. Thus, derivability is a more restrictive property than continuity. In fact, there are functions which are continuous everywhere but differentiable nowhere.

If $Rf'(x_0)$ and $Lf'(x_0)$ are finite (they may or may not be equal), then $f(x)$ is continuous at $x = x_0$.

99. The function $f(x) = \begin{cases} \dfrac{x(e^{1/x} - e^{-1/x})}{e^{1/x} + e^{-1/x}}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$ is

(A) continuous everywhere but not differentiable at $x = 0$
(B) continuous and differentiable everywhere
(C) not continuous at $x = 0$
(D) None of these

100. If $f(x) = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} (\log a)^n$, then at $x = 0$, $f(x)$

(A) has no limit
(B) is continuous
(C) is continuous but not differentiable
(D) is differentiable

101. Let $f(x) = \begin{cases} \int_0^x (5 + |1 - t|)\, dt, & x > 2 \\ 5x + 1 & , x \leq 2 \end{cases}$, then at $x = 2$

(A) $f(x)$ is continuous
(B) $f(x)$ is not continuous
(C) $f(x)$ is differentiable
(D) $f(x)$ is not differentiable

Passage 3

Let f be a function defined on an interval I. If f be discontinuous at a point $p \in I$, then we say that

(i) f has a removable discontinuity at p if $\lim\limits_{x \to p} f(x)$ exists but is not equal to $f(p)$.
(ii) f has a discontinuity of the first kind at p if $\lim\limits_{x \to p-0} f(x)$ and $\lim\limits_{x \to p+0} f(x)$ exist but are unequal.
(iii) f has a discontinuity of the second kind at p if neither of $\lim\limits_{x \to p-0} f(x)$ and $\lim\limits_{x \to p+0} f(x)$ exists.

102. The function $f(x)$ = are tan $\dfrac{1}{x-5}$ has

(A) discontinuity of the first kind at $x = 5$
(B) discontinuity of the second kind at $x = 5$
(C) removable discontinuity at $x = 5$
(D) continuous at $x = 5$.

103. The function $f(x) = \dfrac{1 - u^2}{2 + u^2}$, where $u = \tan x$, has

(A) discontinuity of the first kind at $x = n\pi \pm \dfrac{\pi}{2}$, $n \in I$
(B) discontinuity of the second kind at $x = n\pi \pm \dfrac{\pi}{2}$, $n \in I$

(C) removable discontinuity at $x = n\pi \pm \frac{\pi}{2}$, $n \in I$

(D) continuous at $x = n\pi \pm \frac{\pi}{2}$, $n \in I$

104. The function $f(x) = t^3$, where

$$t = \begin{cases} x-1, & x \le 0 \\ x+1, & 0 < x < 1 \\ 1, & x = 1 \\ 3-x, & 1 < x \end{cases}$$

has

(A) discontinuity of first kind at $x = 0$

(B) removable discontinuity at $x = 1$

(C) discontinuity of first kind at $x = 1$

(D) removable discontinuity at $x = 0$

105. Let $f(x) = \frac{1}{[\cos x]}$, where $[\cdot]$ denotes the greatest integer function. Then, the function $f(x)$ has at $x = \frac{\pi}{2}$

(A) removable discontinuity

(B) discontinuity of first kind from left

(C) discontinuity of second kind from left

(D) None of these

Match the Column Type

106.

Column-I	Column-II
I. Let f and g be differentiable functions satisfying $g'(a) = 2$, $g(a) = b$ and $fog = I$ (identity function). Then, $f'(b) =$	(A) $\frac{2}{3}$
II. Let $f(x) = \begin{cases} (1+\lvert\sin x\rvert)^{a/\lvert\sin x\rvert}, & -\frac{\pi}{6} < x < 0 \\ e^{\tan 2x/\tan 3x}, & 0 < x < \frac{\pi}{6} \\ e^{2/3}, & x = 0 \end{cases}$ The value of a so that $f(x)$ may be continuous at $x = 0$ is	(B) $\frac{1}{12}$
III. The value of $f(\pi/4)$ so that the function $f(x) = (\sin 2x)\tan^2 2x$ is continuous at $x = \pi/4$ is	(C) $\frac{1}{2}$
IV. The value of $f(0)$ so that the function $f(x) = \frac{\sqrt[3]{1+x} - \sqrt[4]{1+x}}{x}$ becomes continuous at $x = 0$, is	(D) $-\frac{1}{2}$

107.

Column-I	Column-II
I. If $f(x)$ is continuous in [0, 1] and $f\left(\frac{1}{2}\right) = 2$, then $\lim_{n\to\infty} f\left(\frac{\sqrt{n}}{2\sqrt{n}+1}\right) =$	(A) 6
II. If a function f, defined and continuous on [2, 5], takes rational values for all x and $f(4) = 8$, then $f(3\cdot7) =$	(B) 2
III. The number of points in the interval (1, 2), where the function $f(x) = [x^3 - 3]$ ([·] denotes the greatest integer function) is discontinuous, is	(C) 3
IV. The number of points of discontinuity of the function $f(x) = [3 + 2\cos x]$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, where [·] denotes the greatest integer function, is	(D) 8

108.

Function	Character of discontinuity
I. $f(x) = \|2\sin 2x\| + 2$ at $x = 0$	(A) Oscillating discontinuity
II. $f(x) = \begin{cases} \tan\dfrac{\pi x}{2}, & x < 1 \\ x - 1, & 1 \le x < 2 \end{cases}$ at $x = 1$	(B) Infinite discontinuity
III. $f(x) = \begin{cases} \sin\dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$	(C) Removable discontinuity
IV. $f(x) = \dfrac{\|x+2\|}{\tan^{-1}(x+2)}$ at $x = -2$	(D) Jump discontinuity

109. The equationshave at least one root on the interval

Equations	Interval
I. $\sin x - x + 1 = 0$	(A) $(-2, 1/2)$
II. $x^2/4 - \sin px + \dfrac{2}{3} = 0$	(B) $(0, 1)$
III. $x^3/4 - \sin px + \dfrac{2}{3} = 0$	(C) $(0, 3\pi/2)$
IV. $2^x - 3x = 0$	(D) $(-2, 2)$

Assertion-Reason Type

Instructions: *In the following questions an Assertion (A) is given followed by a Reason (R). Mark your responses from the following options:*

(A) Assertion(A) is True and Reason(R) is True; Reason(R) is a correct explanation for Assertion(A)

(B) Assertion(A) is True, Reason(R) is True; Reason(R) is not a correct explanation for Assertion(A)

(C) Assertion(A) is True, Reason(R) is False

(D) Assertion(A) is False, Reason(R) is True

110. Assertion: Let $f(x + y) = f(x) f(y)$ for all x, y, where $f(0) \ne 0$. If $f'(0) = 2$, then $f(x) = Ae^{2x}$, where A is a constant.

Reason: $f'(x) = f(x)$

111. Assertion: Let $f: R \to R$ be a function defined by $f(x) = \max.\{x, x^3\}$. Then, $f(x)$ is not differentiable at $x = -1, 0, 1$

Reason: $f(x) = \begin{cases} x, & x \le -1 \\ x^3, & -1 < x \le 0 \\ x, & 0 < x \le 1 \\ x^3, & x > 1 \end{cases}$

112. Assertion: Let $f: R \to R$ be any function. Define $g: R \to R$ by $g(x) = |f(x)|$ for all x. Then, g is continuous if f is continuous.

Reason: Composition of two continuous functions is continuous

113. Assertion: If $f(x) = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right)$, then $f(x)$ is differentiable everywhere

Reason: $f'(x) = \begin{cases} \dfrac{-2}{1+x^2}, & \text{if } |x| < 1 \\ \dfrac{2}{1+x^2}, & \text{if } |x| > 1 \end{cases}$

114. Assertion: The function $f(x) = \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \dfrac{[2rx]}{n^2}$,

where $[\,\cdot\,]$ denotes the greatest integer function, is continuous everywhere.

Reason: $f(x) = x, \forall\, x$

115. **Assertion:** The function $f(x) = \lim_{n\to\infty} \frac{\cos \pi x - x^{2n} \sin(x-1)}{1 + x^{2n+1} - x^{2n}}$ is discontinuous at $x = \pm 1$

Reason: $f(x) = \begin{cases} \frac{\cos \pi x}{1+x}, & |x|<1 \\ -1+\sin 2, & x=-1 \\ -1, & x=1 \\ \frac{-\sin(x-1)}{x-1}, & |x|>1 \end{cases}$

116. **Assertion:** If $f(x) = \text{sgn}(x)$ and $g(x) = x(1-x^2)$, then $fog(x)$ and $gof(x)$ are continuous everywhere

Reason: $fog(x) = \begin{cases} -1, & x \in (-1, 0) \cup (1, \infty) \\ 0, & x \in \{-1, 0, 1\} \\ 1, & x \in (-\infty, -1) \cup (0, 1) \end{cases}$

and, $gof(x) = 0, \forall x \in R$

117. **Assertion:** Let f be a function such that $f(xy) = f(x)\cdot f(y)$, $\forall y \in R$ and $f(1+x) = 1 + x(1 + g(x))$, where $\lim_{x\to 0} g(x) = 0$, then

$$\int_1^2 \frac{f(x)}{f'(x)} \cdot \frac{1}{1+x^2}\, dx = \frac{1}{2}\log\left(\frac{5}{2}\right)$$

Reason: $f'(x) = \frac{f(x)}{x}$

118. **Assertion:** The function $y = f(x)$, defined parametrically as $y = t^2 + t|t|$, $x = 2t - |t|$, $t \in R$, is continuous for all real x.

Reason: $f(x) = \begin{cases} 2x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$

Previous Year's Questions

119. Let $f(x) = \frac{1-\tan x}{4x-\pi}$, $x \neq \frac{\pi}{4}$, $x \in \left[0, \frac{\pi}{2}\right]$. If $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$, then $f\left[\frac{\pi}{4}\right]$ is **[2004]**

(A) 1 (B) $\frac{1}{2}$

(C) $-\frac{1}{2}$ (D) -1

120. Let $f: R \to R$ be a function defined by $f(x) = \min\{x+1, |x|+1\}$. Then which of the following is true? **[2007]**

(A) $f(x) \geq 1$ for all $x \in R$
(B) $f(x)$ is not differentiable at $x = 1$
(C) $f(x)$ is differentiable everywhere
(D) $f(x)$ is not differentiable at $x = 0$

121. The function $f: R \sim \{0\} \to R$ given by

$$f(x) = \frac{1}{x} - \frac{2}{e^{2x}-1}$$

can be made continuous at $x = 0$ by defining $f(0)$ as **[2007]**

(A) 2 (B) -1

(C) 0 (D) 1

122. Let $f(x) = \begin{cases} (x-1)\sin\left(\frac{1}{x-1}\right) & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$ Then which one of the following is true? **[2008]**

(A) f is neither differentiable at $x = 0$ nor at $x = 1$
(B) f is differentiable at $x = 0$ and at $x = 1$
(C) f is differentiable at $x = 0$ but not at $x = 1$
(D) f is differentiable at $x = 1$ but not at $x = 0$

123. Consider the following relations:

$R = \{(x, y) | x, y$ are real numbers and $x = wy$ for some rational number $w\}$;.

$S = \left\{\left(\frac{m}{p}, \frac{p}{q}\right) \middle| m, n, p \text{ and } q \in Z \text{ such that } n, q \neq 0 \text{ and } qm = pn\right\}$ Then **[2010]**

(A) neither R nor S is an equivalence relation
(B) S is an equivalence relation but R is not an equivalence relation
(C) R and S both are equivalence relations
(D) R is an equivalence relation but S is not an equivalence relation

124. The real values of p and q for which the function

$$f(x)=\begin{cases}\dfrac{\sin(p+1)x+\sin x}{x}, x<0\\ q, x=0\\ \dfrac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}}, x>0\end{cases}$$ is continuous for all x in R, is **[2011]**

(A) $p=\frac{5}{2}, q=\frac{1}{2}$ (B) $p=-\frac{3}{2}, q=\frac{1}{2}$

(C) $p=\frac{1}{2}, q=\frac{3}{2}$ (D) $p=\frac{1}{2}, q=-\frac{3}{2}$

125. If $f: R \to R$ is a function defined by $f(x)=[x]\cos\left(\frac{2x-1}{2}\right)\pi$, where $[x]$ denotes the greatest integer function, then f is **[2012]**

(A) continuous for every real x
(B) discontinuous only at $x = 0$
(C) discontinuous only at non-zero integral values of x
(D) continuous only at $x = 0$

126. Consider the function $f(x)=|x-2|+|x-5|, x \in R$. **[2012]**

Statement 1: $f'(4)=0$

Statement 2: f is continuous in [2, 5], differentiable in (2, 5) and $f(2) = f(5)$.

(A) Statement 1 is false, statement 2 is true
(B) Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1
(C) Statement 1 is true, statement 2 is true; statement 2 is not a correct explanation for statement 1
(D) Statement 1 is true, statement 2 is false

127. If the function $g(x)=\begin{cases}k\sqrt{x+1}, & 0\le x\le 3\\ mx+2, & 3<x\le 5\end{cases}$ is differentiable, then the value of $k + m$ is: **[2015]**

(A) $\frac{16}{5}$ (B) $\frac{10}{3}$

(C) 4 (D) 2

128. For $x \in R, f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then: **[2016]**

(A) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$
(B) g is not differentiable at $x = 0$
(C) $g'(0) = \cos(\log 2)$
(D) $g'(0) = -\cos(\log 2)$

ANSWER KEYS

Single Option Correct Type

1. (C)	**2.** (D)	**3.** (B)	**4.** (B)	**5.** (C)
6. (A)	**7.** (B)	**8.** (A, B, D)	**9.** (D)	**10.** (C)
11. (A)	**12.** (B)	**13.** (B)	**14.** (A)	**15.** (C)
16. (B, C)	**17.** (A)	**18.** (A, C)	**19.** (A, B)	**20.** (A, B)
21. (D)	**22.** (B)	**23.** (B)	**24.** (C)	**25.** (A)
26. (B)	**27.** (C)	**28.** (B)	**29.** (C)	**30.** (A)
31. (B)	**32.** (D)	**33.** (A)	**34.** (B)	**35.** (B)
36. (B)	**37.** (C)	**38.** (D)	**39.** (A)	**40.** (B)
41. (D)	**42.** (A)	**43.** (C)	**44.** (D)	**45.** (D)
46. (A)	**47.** (A)	**48.** (C)	**49.** (A)	**50.** (D)
51. (C)	**52.** (C)	**53.** (C)	**54.** (C)	**55.** (C)
56. (A)	**57.** (D)	**58.** (B)	**59.** (C)	**60.** (B)
61. (D)	**62.** (D)	**63.** (A)	**64.** (C)	**65.** (A)
66. (C)	**67.** (B)	**68.** (C)	**69.** (B)	**70.** (A)
71. (A)	**72.** (D)	**73.** (A)	**74.** (B)	**75.** (C)
76. (A)				

More than One Option Correct Type

77. (A, B) **78.** (A, B) **79.** (A, C) **80.** (B, C) **81.** (B, C)
82. (A, B, D) **83.** (A, D) **84.** (B, C) **85.** (B, D) **86.** (A, B)
87. (A, D) **88.** (A, B, D) **89.** (B, C) **90.** (A, C) **91.** (A, D)
92. (A, D) **93.** (A, C) **94.** (C, D) **95.** (A, B, C, D)

Passage Based Questions

96. (C) **97.** (D) **98.** (A) **99.** (A) **100.** (B, D) **101.** (A, D) **102.** (A)
103. (C) **104.** (A, B) **105.** (C)

Match the Column Type

106. I $\rightarrow$ (C), II $\rightarrow$ (A), III $\rightarrow$ (D), IV $\rightarrow$ (B), **107.** I $\rightarrow$ (B), II $\rightarrow$ (D), III $\rightarrow$ (A), IV $\rightarrow$ (C)
108. I $\rightarrow$ (C), II $\rightarrow$ (B), III $\rightarrow$ (A), IV $\rightarrow$ (D) **109.** I $\rightarrow$ (C), II $\rightarrow$ (A), III $\rightarrow$ (D), IV $\rightarrow$ (B)

Assertion-Reason Type

110. (C) **111.** (A) **112.** (A) **113.** (D) **114.** (A)
115. (A) **116.** (D) **117.** (A) **118.** (A)

Previous Year's Questions

119. (C) **120.** (C) **121.** (D) **122.** (A) **123.** (A) **124.** (B) **125.** (A) **126.** (B) **127.** (D) **128.** (C)

HINTS AND SOLUTIONS

Single Option Correct Type

1. We have,

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{3-1}{-h} \to -\infty$$

$\therefore$ $f(x)$ is not differentiable at $x = 0$

Also, if $x < 0$ or $x \geq 0$ then $|x| \geq 0$

$\therefore$ $f(|x|) = 2|x| + 1$ for all x.

$$\therefore\ Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{2h+1-1}{h} = 2$$

$$\text{and } Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{2(-h)+1-1}{-h} = 2$$

$\therefore$ $f(|x|)$ is differentiable at $x = 0$.

The correct option is (C)

2. $L(gof)'\left(\frac{\pi}{2}\right)$

$$= \lim_{h\to 0} \frac{(gof)\left(\frac{\pi}{2}-h\right)-(gof)\left(\frac{\pi}{2}\right)}{-h}$$

$$= \lim_{h\to 0} \frac{\left[\cos\left(\frac{\pi}{2}-h\right)+2\right]-\left[\cos\frac{\pi}{2}+2\right]}{-h}$$

$$= \lim_{h\to 0} \frac{(\sin h+2)-(2)}{-h} = \lim_{h\to 0} \frac{2-2}{-h} = 0$$

$$R(gof)'\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{(gof)\left(\frac{\pi}{2}+h\right)-(gof)\left(\frac{\pi}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left[\cos\left(\frac{\pi}{2}+h\right)+2\right]-\left[\cos\frac{\pi}{2}+2\right]}{h}$$

$$= \lim_{h\to 0} \frac{(-\sin h+2)-(2)}{h}$$

$$= \lim_{h\to 0} \frac{1-2}{h} \to -\infty$$

$\therefore$ (gof) is not differentiable at $x = \pi/2$.

The correct option is (D)

3. The given function is clearly continuous at all points except possibly at $x = \pm 1$.

For $f(x)$ to be continuous at $x = 1$, we must have

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{x\to 1} ax^2 + b = \lim_{x\to 1} \frac{1}{|x|}$$

$$\Rightarrow a + b = 1 \qquad (1)$$

Now, for $f(x)$ to be differentiable at $x = 1$, we must have

$$\lim_{x\to 1-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1+} \frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow \lim_{x\to 1} \frac{ax^2+b-1}{x-1} = \lim_{x\to 1} \frac{\frac{1}{|x|}-1}{x-1}$$

$$\Rightarrow \lim_{x\to 1} \frac{ax^2-a}{x-1} = \lim_{x\to 1-} \frac{\frac{1}{|x|}-1}{x-1}$$

$$(\because a+b=1 \therefore b-1=-a)$$

$$\Rightarrow \lim_{x\to 1} a(x+1) = \lim_{x\to 1} \frac{-1}{x} \Rightarrow 2a = -1$$

$$\Rightarrow a = -\frac{1}{2}$$

Putting $a = -\frac{1}{2}$ in (1), we get $b = \frac{3}{2}$.

The correct option is (B)

4. We have,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)\cdot f(h)-f(x)}{h}$$

[Using $f(x+y) = f(x)\cdot f(y)$]

$$= f(x) \lim_{h\to 0} \frac{f(h)-1}{h}$$

$$= f(x) \lim_{h\to 0} \frac{1+h\,\phi(h)+h^2\,\phi(h)\,\psi(h)-1}{h}$$

$$= f(x) \lim_{h\to 0} [\phi(h)+h\,\phi(h)\,\psi(h)]$$

$$= f(x)\,(a + 0\cdot a\cdot b) = a\cdot f(x).$$

The correct option is (B)

5. For $n \in I$,

$$\lim_{x\to n^+} f(x) = \lim_{x\to n^+} [x]\cos\frac{2x-1}{2}\pi$$
$$= n\cos\frac{2n-1}{2}\pi = 0$$

and $$\lim_{x\to n^-} f(x) = \lim_{x\to n^-} [x]\cos\frac{2x-1}{2}\pi$$
$$= (n-1)\cos\frac{2n-1}{2}\pi = 0.$$

Hence, f is continuous for $x = n \in I$. Since the functions $g(x) = [x]$ and $h(x) = \cos\frac{2x-1}{2}\pi$ are continuous on $x \in R - I$, so f is continuous everywhere.

The correct option is (C)

6. $$f'(k-0) = \lim_{h\to 0}\frac{[k-h]\sin\pi(k-h)-[k]\sin\pi k}{-h}$$
$$= \lim_{h\to 0}\frac{(-1)^{k-1}(k-1)\sin\pi h - k\times 0}{-h}$$
$$= \lim_{h\to 0}\frac{(-1)^{k-1}\cdot(k-1)\sin\pi h}{-h}$$
$$= \mathbf{(-1)^k \cdot (k-1)\,\pi.}$$

The correct option is (A)

7. $$\lim_{x\to a-} f(x)\cdot g(x) = \lim_{x\to a-} f(x)\cdot \lim_{x\to a-} g(x) = ml$$

and $$\lim_{x\to a+} f(x)\cdot g(x) = \lim_{x\to a+} f(x)\cdot \lim_{x\to a+} g(x) = lm$$

$$\therefore \quad \lim_{x\to a} f(x)\cdot g(x) = lm$$

The correct option is (B)

8. We have $\lim_{h\to 0} f(0-h) = \lim_{h\to 0}[-h\sin(-\pi h)]$

$$= \lim_{h\to 0}[h\sin\pi h] = 0,$$

$\lim_{h\to 0} f(0+h) = \lim_{h\to 0}[h\sin\pi h] = 0.$ and $f(0) = 0$

$\therefore f(x)$ is continuous at $x = 0$.

It can be easily seen that **$f(x)$ is continuous in $(-1, 0)$.** $f(x)$ is not differentiable at $x = 1$ but **it is differentiable in $(-1, 1)$.**

The correct option is (A), (B), and (D.)

9. Note that $f(x) = 0$ for each integral value of x.

Also, if $0 \le x < 1$, then $0 \le x^2 < 1$

$\therefore \quad [x] = 0$ and $[x^2] = 0 \Rightarrow f(x) = 0$ for $0 \le x < 1$

Next, if $1 \le x < \sqrt{2}$, then

$1 \le x^2 < 2 \Rightarrow [x] = 1$ and $[x^2] = 1$

Thus, $f(x) = [x]^2 - [x^2] = 0$ if $1 \le x < \sqrt{2}$.

It follows that $f(x) = 0$ if $0 \le x < \sqrt{2}$.

This shows that $f(x)$ must be continuous at $x = 1$.

However, at points x other than integers and not lying between 0 and $\sqrt{2}$, $f(x) \ne 0$.

Thus, **f is discontinuous at all integers except 1.**

The correct option is (D)

10. Let $h(x) = |x|$ for all x. Clearly, $h(x)$ is continuous for all x. Then $g(x) = |f(x)| = h[f(x)] = (hof)(x)$ for all x.
Since composition of two continuous functions is continuous, therefore, **g is continuous if f is continuous.**
The correct option is (C)

11. Since $f'(0)$ exists,

$$\therefore \quad Rf'(0) = Lf'(0)$$

$$\Rightarrow \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$
$$= -\lim_{h\to 0}\frac{f(-h)-f(0)}{h}$$
$$= -\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$[\because f(-h) = f(h)]$

$$\Rightarrow 2\lim_{h\to 0}\frac{f(h)-f(0)}{h} = 0 \Rightarrow 2f'(0) = 0$$

$[\because f'(0)$ exists$]$

$$\Rightarrow \quad f'(0) = 0$$

The correct option is (A)

12. We have,

$$Lf'(0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$
$$= \lim_{h\to 0}\left[\frac{-h(3e^{-1/h}+4)}{2-e^{-1/h}} - 0\right]\cdot\left(\frac{-1}{h}\right)$$
$$= \frac{0+4}{2-0} = 2$$

$$Rf'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$
$$= \lim_{h\to 0}\left[\frac{h(3e^{1/h}+4)}{2-e^{1/h}} - 0\right]\frac{1}{h}$$
$$= \lim_{h\to 0}\left(\frac{3+4e^{-1/h}}{2e^{-1/h}-1}\right) = \frac{3+0}{0-1} = -3$$

Since $Lf'(0) \ne Rf'(0)$, **$\therefore f'(x)$ is not differentiable at $x = 0$.** But **$f(x)$ is continuous at $x = 0$** (as $Lf'(0)$ and $Rf'(0)$ are finite).

The correct option is (B)

13. The function $u = \dfrac{1}{x-1}$ suffers a discontinuity at the point $x = 1$.

The function $f(x) = \dfrac{1}{u^2 + u - 2}$ suffers a discontinuity at the points where $u^2 + u - 2 = 0$ i.e., $u = -2$ and $u = 1$. Using these values of u, the corresponding values of x are obtained by solving the equations

$$-2 = \frac{1}{x-1} \text{ and } 1 = \frac{1}{x-1} \text{ i.e. } x = 1/2 \text{ and } x = 2.$$

Hence, the composite function is discontinuous at three points $x = 1/2$, $x = 1$ and $x = 2$.

The correct option is (B)

14. $3 \le 3 + 2\cos x \le 5$ for $x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

$f(x) = [3 + 2\cos x]$ is discontinuous at those points where $3 + 2\cos x$ is an integer.

Now, $3 + 2\cos x = 3$ if $\cos x = 0$. So, $x = \dfrac{-\pi}{2}, \dfrac{\pi}{2}$ (not possible)

$3 + 2\cos x = 4$ if $\cos x = \dfrac{1}{2}$

So x has two values $\dfrac{\pi}{3}$ and $\dfrac{-\pi}{3}$.

$3 + 2\cos x = 5$ if $\cos x = 1$. So, $x = 0$.

$\therefore$ The number of values of $x = 2 + 1 = 3$.

The correct option is (A)

15. The function $f(x)$ is continuous at all points where

$$\frac{1}{2} - \cos^2 x \ge 0 \Rightarrow |\cos x| \le \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{\pi}{4} + 2n\pi \le x \le \frac{3\pi}{4} + 2n\pi$$

$$\text{or} \quad \frac{5\pi}{4} + 2n\pi \le x \le \frac{7\pi}{4} + 2n\pi,\ n \in I.$$

The correct option is (C)

16. Clearly, $x = 1$ is a point of discontinuity of the function $f(x) = \dfrac{1}{1-x}$.

If $x \ne 1$, then $(fof)(x) = f[f(x)] = f\left(\dfrac{1}{1-x}\right) = \dfrac{x-1}{x}$, which is discontinuous at $x = 0$.

If $x \ne 0$ and $x \ne 1$, then

$$(fofof)(x) = f[(fof)(x)] = f\left(\frac{x-1}{x}\right) = x,$$

which is continuous everywhere.

Hence, $f^{3n}(x) = (fofof)^n(x) = x$, which is continuous everywhere.

So, the only points of discontinuity are $x = 0$ and $x = 1$.

The correct option is (B) and (C)

17. We have,

$$\lim_{h \to 0} f(5-h) = \lim_{h \to 0} \tan^{-1}\frac{1}{(5-h)-5}$$
$$= \lim_{h \to 0} \tan^{-1}\left(\frac{-1}{h}\right)$$
$$= \tan^{-1}(-\infty) = \frac{-\pi}{2}$$

and
$$\lim_{h \to 0} f(5+h) = \lim_{h \to 0} \tan^{-1}\frac{1}{(5+h)-5}$$
$$= \lim_{h \to 0} \tan^{-1}\left(\frac{1}{h}\right)$$
$$= \tan^{-1}(\infty) = \frac{\pi}{2}$$

Since $\lim_{h \to 0} f(5-h) \ne \lim_{h \to 0} f(5+h)$, therefore, $f(x)$ has **discontinuity of the first kind at $x = 5$.**

The correct option is (A)

18. Since $|x-1|, |x-1|^2$, etc, are continuous at $x = 1$ $\therefore f(x)$ is continuous at $x = 1$ for all $a_k \in R$.

Also, $|x-1|^2, |x-1|^4$, etc, are all differentiable at $x = 1$, whereas $|x-1|, |x-1|^3$, etc, are not differentiable at $x = 1$. Therefore, $f(x)$ is differentiable at $x = 1$ for all $a_{2k+1} = 0$.

The correct option is (A) and (C)

19. $$f'(x) = \frac{-1}{\sqrt{1-\left(\dfrac{2x}{1+x^2}\right)^2}} \times \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$= \frac{-(1+x^2)}{\sqrt{(1+x^2)^2 - 4x^2}} \times \frac{2(1-x^2)}{(1+x^2)^2}$$

$$= \frac{-2}{1+x^2} \cdot \frac{1-x^2}{|1-x^2|} = \begin{cases} \dfrac{-2}{1+x^2}, & \text{if } |x| < 1 \\ \dfrac{2}{1+x^2}, & \text{if } |x| > 1 \end{cases}$$

Clearly, $f(x)$ is differentiable everywhere except at the points where $|x| = 1$ i.e. $x = \pm 1$.

Hence, $f(x)$ is differentiable on $(-\infty, \infty) \therefore \{-1, 1\}$.

The correct option is (C)

20. The function $[\sqrt{x}]$ has discontinuity at every $x = n^2$ and $\sqrt{x}$ is not defined on $(-\infty, 0)$. Hence the set of discontinuities of $f(x)$ is $(-\infty, 0) \cup \{n^2 : n \in N\}$. This clearly contains the sets in (a) and (b).

$\therefore$ (a) and (b) are the correct answers.

The correct option is (A, B)

21. We have

$$\lim_{h \to 0} f(3-h) = \lim_{h \to 0} |3-(3-h)| + (3+3-h)$$
$$= \lim_{h \to 0} (h+6) = 6,$$

$$\lim_{h \to 0} f(3+h) = \lim_{h \to 0} |3-(3+h)| + (3+3+h)$$

$$= \lim_{h \to 0} (h+7) = 7$$

Since $\lim_{h \to 0} f(3-h) \neq \lim_{h \to 0} f(3+h)$, therefore, $f(x)$ is not continuous and hence not differentiable at $x = 3$.

The correct option is (D)

22. Since $f(x)$ is continuous at $x = \pi$

$\therefore \quad \lim_{x \to \pi} f(x) = f(\pi) \Rightarrow \lim_{h \to 0} f(\pi - h) = k$

$$\Rightarrow \quad k = \lim_{h \to 0} \frac{1+\cos(\pi - h)}{(\pi - \pi + h)^2} \cdot \frac{\sin^2(\pi - h)}{\log[1+\pi^2 - 2\pi(\pi - h) + (\pi - h)^2]}$$

$$= \lim_{h \to 0} \frac{1-\cos h}{h^2} \cdot \frac{\sin^2 h}{\log(1+h^2)}$$

$$= \lim_{h \to 0} \frac{1}{2} \cdot \left(\frac{\sin h/2}{h/2}\right)^2 \cdot \frac{h^2}{\log(1+h^2)} \cdot \left(\frac{\sin h}{h}\right)^2$$

$$= \frac{1}{2}. \therefore k = 1/2$$

The correct option is (B)

23. A continuous function $f(x)$ defined on $1 \le x \le 3$ having only rational values must be a constant function.

$\therefore \quad f(2) = 10 \Rightarrow f(x) = 10$, for all x

$\therefore \quad f(1.5) = 10$

The correct option is (B)

24. We have, $f(x) = \int_0^x t \cos\frac{1}{t}\, dt \Rightarrow f'(x) = x \cos\frac{1}{x}$.

Clearly $f'(x)$ exists and is finite in the interval $(0, \pi)$. Therefore, $f(x)$ is differentiable in the interval $(0, \pi)$.

Hence, $f(x)$ is continuous in the interval $(0, \pi)$.

The correct option is (C)

25. Let $x^3 = n, n \in \mathrm{I} \Rightarrow x = n^{1/3}$

$\therefore \quad f(x) = (-1)^n = \pm 1$

Hence, $f(x)$ is discontinuous for $x = n^{1/3}$, $n \in I$.

The correct option is (A)

26. We have, $f(x) = \left[\frac{1}{\sqrt{2}}(\cos x + \sin x)\right]$

$$= \left[\cos\left(x - \frac{\pi}{4}\right)\right]$$

Clearly, $f(x)$ is discontinuous at all those points where $\cos\left(x - \frac{\pi}{4}\right)$ is an integer *i.e.* $x - \frac{\pi}{4} = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

i.e., $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$.

The correct option is (B)

27. Since $[x]$ and $e^{|x|}$ are not differentiable at $x = 0$, therefore, for $f(x)$ to be differentiable at $x = 0$, we must have $\boldsymbol{a = 0, b = 0}$ **and c can be any real number.**

The correct option is (C)

28. $[x+1] = 0$ if $0 \le x+1 < 1$ *i.e.* $-1 \le x < 0$.

Thus domain of $f = R \therefore [-1, 0)$.

We have, $\sin\left(\frac{\pi}{[x+1]}\right)$ continuous at all points of $R \therefore [-1, 0)$ and $[x]$ continuous on $R \setminus Z$, where Z denotes the set of integers. Thus the points where f can possibly be discontinuous are ..., $-3, -2, -1, 0, 1, 2$, ...

For $0 \le x < 1$, $[x] = 0$ and $\sin\left(\frac{\pi}{[x+1]}\right)$ is defined.

$\therefore \quad f(x) = 0$ for $0 \le x < 1$.

Also, f is not defined on $[-1, 0)$, so the continuity of f at 0 means continuity of f from right at 0. Since f is continuous from right at 0, so f is continuous at 0. Hence the set of points of discontinuity of f is $Z \setminus \{0\}$.

The correct option is (B)

29. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x) + f(h) - f(x)}{h} \qquad [\because f(x+y) = f(x) + f(y)]$$

$$= \lim_{h \to 0} \frac{h^3 \phi(h)}{h} \qquad [\because f(x) = x^3 \phi(x)]$$

$$= h^2 \phi(h) = 0 \times \phi(0)$$

$[\phi$ is continuous at $x = 0$, $\therefore \lim_{h \to 0} \phi(h) = \phi(0)]$

$= 0$

The correct option is (C)

30. For $f(x)$ to be continuous at $x = 0$, we must have

$$f(0) = \lim_{x \to 0} \frac{\sqrt[3]{1+x} - \sqrt[4]{1+x}}{x}$$

$$= \lim_{x \to 0} \frac{\left[1 + \frac{1}{3}x + \frac{(1/3)(-2/3)}{2!}x^2 + \ldots\right]}{x} - \frac{\left[1 + \frac{1}{4}x + \frac{(1/4)(-3/4)}{2!}x^2 + \ldots\right]}{x}$$

$$= \lim_{x \to 0} \frac{x\left[\frac{1}{12} + \left(\frac{-1}{9} + \frac{3}{32}\right)x + \text{terms containing } x^2 \text{ and higher powers}\right]}{x}$$

$$= \frac{1}{12} \quad \therefore f(0) = \frac{1}{12}$$

The correct option is (A)

31. Let $f'(0+) = \lim_{h\to 0} \dfrac{f(h)-f(0)}{h} = k$ (say)

$\therefore \quad f'(0-) = \lim_{h\to 0} \dfrac{f(0)-f(0-h)}{h}$

$= \lim_{h\to 0} \dfrac{f(0)-f(h)}{h} = -k$

$\therefore \quad f'(0^+) \ne f'(0^-)$, but both are finite

so **$f'(x)$ is continuous at $x = 0$ but not differentiable at $x = 0$.**

The correct option is (B)

32. We have, $f(0) = 0$, $g(x) = \dfrac{f(x)}{x}$,

$\lim_{x\to 0} g(x) = \lim_{x\to 0} \dfrac{f(x)}{x} = \lim_{x\to 0} f'(x) = f'(0)$

Thus the required value is **$f'(0)$.**

The correct option is (D)

33. Since $f(x) = \dfrac{1}{[\sin x]}$, $\therefore$ $\sin x \notin [0, 1)$

$\Rightarrow x \notin [2n\pi, (2n+1)p] - (4n+1)\pi/2, n \in I.$

$\therefore$ **$f(x)$ is not continuous if $x \in (2n\pi, 2n\pi + \pi), n \in I$.**

The correct option is (A)

34. For $x \ne -1$, we have

$f(x) = 1 - 2x + 3x^2 - 4x^3 + \ldots\infty$

$= (1+x)^{-1} = \dfrac{1}{1+x}.$

$\lim_{h\to 0} f(-1-h) = \lim_{h\to 0} \dfrac{1}{1-1-h} \to -\infty$

So, $f(x)$ is not continuous at $x = -1$.

Also, $\lim_{h\to 0} \dfrac{f(-1-h) - f(-1)}{(-1-h)-(-1)} = \lim_{h\to 0} \dfrac{\frac{-1}{h} - 1}{-h}$

$= \lim_{h\to 0} \dfrac{1+h}{h^2} \to \infty.$

So, $f(x)$ is not derivable at $x = -1$.

Hence, **$f(x)$ is neither continuous nor derivable at $x = -1$.**

The correct option is (B)

35. For $x \ne 0$, we have

$f(x) = x + \dfrac{x/1+x}{1 - \dfrac{1}{1+x}} = x + \dfrac{x/1+x}{x/1+x} = x + 1.$

For $x = 0, f(x) = 0$

Thus, $f(x) = \begin{cases} x+1, & x \ne 0 \\ 0, & x = 0 \end{cases}$

Clearly, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 1 \ne f(0).$

So, $f(x)$ is discontinuous and hence not differentiable at $x = 0$.

The correct option is (B)

36. Continuity at $x = 2$:

L.H.L. $= \lim_{h\to 0} f(2-h) = \lim_{h\to 0} [(2-h)^2] + [-2+h]^2$

$= \lim_{h\to 0} \{3 + (-2)^2\} = 7.$

R.H.L. $= \lim_{h\to 0} f(2+h) = \lim_{h\to 0} [(2+h)^2] + [-2-h]^2$

$= \lim_{h\to 0} \{4 + (-3)^2\} = 13.$

Since L.H.L. $\ne$ R.H.L., $\therefore f(x)$ is not continuous at $x = 2$.

Differentiability at $x = 2$:

LHD $= \lim_{h\to 0} \dfrac{f(2-h) - f(2)}{(2-h)-2} = \lim_{h\to 0} \dfrac{7-8}{-h} \to \infty.$

So, $f(x)$ is not differentiable at $x = 2$.

Hence, the **function $f(x)$ is neither continuous nor derivable at $x = 2$.**

The correct option is (B)

37. Since the function $f(x)$ is continuous at $x = 0$, therefore,

$\lim_{h\to 0} f(0-h) = f(0) = \lim_{h\to 0} f(0+h)$

$\Rightarrow \quad \lim_{h\to 0} (1 - |\tan h|)^{\frac{a}{|\tan h|}} = b = \lim_{h\to 0} e^{\frac{\sin 3h}{\sin 2h}}$

$\Rightarrow \quad \lim_{h\to 0} [(1 - |\tan h|)^{-1/|\tan h|}]^{-a} = b = \lim_{h\to 0} e^{\frac{\sin 3h/3h}{\sin 2h/2h}\cdot\frac{3}{2}}$

$\Rightarrow \quad e^{-a} = b = e^{3/2} \Rightarrow a = \dfrac{-3}{2}$ and $b = e^{3/2}$.

The correct option is (C)

38. We have

$f(x) = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} (\log a)^n = \sum_{n=0}^{\infty} \dfrac{(x \log a)^n}{n!}$

$= e^{x \log a} = e^{\log a^x} = a^x.$

$Lf'(0) = \lim_{h\to 0} \dfrac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \dfrac{a^{-h} - 1}{-h}$

$= \log_e a$

$Rf'(0) = \lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \dfrac{a^h - 1}{h}$

$= \log_e a$

Since $Lf'(0) = Rf'(0)$,

$\therefore \quad f(x)$ is differentiable at $x = 0$.

Since every differentiable function is continuous, therefore, $f(x)$ is continuous at $x = 0$.

The correct option is (D)

39. Since $f(x)$ is continuous for all x, therefore, it is continuous at $x = 1$ also.

$\therefore \quad f(1) = \lim_{h\to 0} f(1-h) \Rightarrow 1 = \lim_{h\to 0} [a(1-h)^2 + b]$

$\Rightarrow \quad a + b = 1 \qquad (1)$

Also, $f(x)$ is differentiable at $x = 1$

$$\Rightarrow \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{a(1-h)^2+b-1}{-h} = \lim_{h\to 0} \frac{\frac{1}{|1+h|}-1}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{(a+b-1)+(h^2-2h)\,a}{-h} = \lim_{h\to 0} \frac{1-1-h}{h(1+h)}$$

$\Rightarrow \; 2a = -1$ (Using $a + b = 1$)

$\therefore \quad a = \frac{-1}{2}$

Hence, $a + b = 1 \Rightarrow b = 1 - a = \frac{3}{2}$.

The correct option is (A)

40. If $x \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$ then $f(x) = [\tan^2 x] = 0$ which is continuous and differentiable at $x = 0$.

The correct option is (B)

41. The given function f is continuous at $x = 0$ if $\lim_{h\to 0} f(0-h)$ $= f(0) = \lim_{h\to 0} f(0+h)$

$\Rightarrow \; p + 2 = q = \frac{1}{2}$

$\Rightarrow \quad p = -\frac{3}{2}, q = \frac{1}{2}$

The correct option is (D)

42. Doubtful points are $x = n, n \in I$

$$\text{L.H.L} = \lim_{x\to n^-} [x]\cos\left(\frac{2x-1}{2}\right)\pi$$

$$= (n-1)\cos\left(\frac{2n-1}{2}\right)\pi = 0$$

$$\text{R.H.L} = \lim_{x\to n^+} [x]\cos\left(\frac{2n-1}{2}\right)\pi$$

$$= n\cos\left(\frac{2n-1}{2}\right)\pi = 0$$

$f(n) = 0$

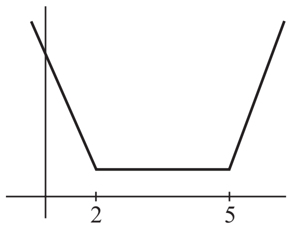
Hence continuous for all real x.

The correct option is (A)

43. $f(x) = 3 \;\; 2 \le x \le 5$

$f'(x) = 0 \;\; 2 < x < 5$

$f'(4) = 0$



The correct option is (C)

44. At $x = 0$ and $1, f(x) = |x| + [x - 1]$

= continuous + discontinuous

= discontinuous

The correct option is (D)

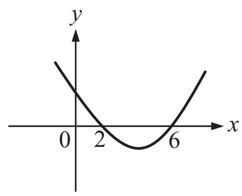
45. RHD of $\sin(|x|) - |x|$ at $x = 0$ is $1 - 1 = 0$

LHD of $\sin(|x|) - |x|$ at $x = 0$ is $(-1) - (-1) = 0$,

so differentiable at $x = 0$

The correct option is (D)

46.



Statement-2 is true, all conditions of Rolle's theorem are satisfied so $f'(c) = 0$ for some $c \in (2, 6)$.

The correct option is (A)

47. $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$

$$\Rightarrow f'(1) = \lim_{h\to 0} \frac{(1+h-1)\sin\left(\frac{1}{1+h-1}\right)-0}{h}$$

$$= \lim_{h\to 0} \frac{h}{h}\sin\left(\frac{1}{h}\right)$$

$$\Rightarrow f'(1) = \lim_{h\to 0} \sin\left(\frac{1}{h}\right)$$

$\therefore f$ is not differentiable at $x = 1$.

Similarly, $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$

$$\Rightarrow f'(0) = \lim_{h\to 0} \frac{(h-1)\sin\left(\frac{1}{h-1}\right)-\sin(1)}{h}$$

$\Rightarrow f$ is also not differentiable at $x = 0$.

The correct option is (A)

48. $f(x) = x\,|x|$ and $g(x) = \sin x$

$$gof(x) = \sin(x\,|x|) = \begin{cases} -\sin x^2, & x < 0 \\ \sin x^2, & x \ge 0 \end{cases}$$

$\therefore\ (\text{gof})'(x) = \begin{cases} -2x\cos x^2, & x<0 \\ 2x\cos x^2, & x\ge 0 \end{cases}$

Clearly, $L(\text{gof})'(0) = 0 = R(\text{gof})'(0)$

$\therefore$ gof is differentiable at $x = 0$ and also its derivative is continuous at $x = 0$

Now, $(\text{gof})''(x) = \begin{cases} -2\cos x^2 + 4x^2\sin x^2, & x<0 \\ 2\cos x^2 - 4x^2\sin x^2, & x\ge 0 \end{cases}$

$\therefore L(\text{gof})''(0) = -2$ and $R(\text{gof})''(0) = 2$

$\therefore L(\text{gof})''(0) \ne R(\text{gof})''(0)$

$\therefore$ gof(x) is not twice differentiable at $x = 0$.

The correct option is (C)

49. The only doubtful point is $x = 0$.

$$\text{L.H.L.} = \lim_{h\to 0^-} f(0-h) = \lim_{h\to 0}\,(-h+1)^{2-\left(\frac{1}{h}-\frac{1}{h}\right)}$$

$$= \lim_{h\to 0}(1-h)^2 = 1$$

$$\text{R.H.L.} = \lim_{h\to 0^-} f(0+h) = \lim_{h\to 0}\,(h+1)^{2-\left(\frac{1}{h}+\frac{1}{h}\right)}$$

$$= \lim_{h\to 0}(1+h)^{2-\frac{2}{h}} = \lim_{h\to 0}(1+h)^2\,[(1+h)^{1/h}]^{-2}$$

$$= 1\times e^{-2} = e^{-2}$$

Since L.H.L $\ne$ R.H.L, $\therefore f(x)$ is not continuous at $x = 0$.

The correct option is (A)

50. $f(x) = \begin{cases} \dfrac{e^{[x]+|x|}-2}{[x]+|x|}, & x\ne 0 \\ -1, & x=0 \end{cases}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{e^{[x]+|x|}-2}{[x]+|x|} = \frac{e^{-1}-2}{-1}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{e^{[x]+|x|}-2}{[x]+|x|}$$

$$= \lim_{x\to 0^+}\frac{e^x-2}{x} \to -\infty$$

The correct option is (D)

51. Let x_0 be any arbitrary real number.

Case I: x_0 is rational

Then, $f(x_0) = 1$

In any vicinity of a rational point there are irrational points, where $f(x) = 0$. Hence, in any vicinity of x_0 there are points x for which

$|\Delta y| = |f(x_0) - f(x)| = 1$

Case II: x_0 is irrational

Then, $f(x_0) = 0$

In any vicinity of an irrational point there are rational points at which $f(x) = 1$. Hence, it is possible to find the values of x for which

$|\Delta y| = |f(x_0) - f(x)| = 1$

Thus, in both cases, the difference Δy does not tend to zero as $\Delta x \to 0$. Therefore, x_0 is a point of discontinuity. Since x_0 is an arbitrary point, the Dirichlet function ***f(x)* is discontinuous at each point.**

The correct option is (C)

52. We have,

$$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3},\ f(0) = 0 \text{ and } f'(0) = 3$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(\frac{3x+3h}{3}\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{f(3x)+f(3h)}{3}-\frac{f(3x)+f(0)}{3}}{h}$$

$$= \lim_{h\to 0}\frac{f(3h)-f(0)}{3h} = 3$$

$\therefore\quad f(x) = 3x + c,\ \therefore f(0) = 0 \Rightarrow c = 0$

$\therefore\quad f(x) = 3x$

The correct option is (C)

53. Let a be any real number

$\lim_{x\to a} f(x) = \lim_{x\to a} x = a$

(when $x \to a$ through rational values)

$\lim_{x\to a} f(x) = \lim_{x\to a}(1-x) = 1-a$

(when $x \to a$ through irrational values)

$\lim_{x\to a} f(x)$ will exist only when $a = 1-a$ or $a = \frac{1}{2}$

Thus, if $x \ne \frac{1}{2}$, then $\lim_{x\to a} f(x)$ will not exist and hence $f(x)$ will be discontinuous at $x = a$ where $a \ne \frac{1}{2}$

Also, $\lim_{x\to\frac{1}{2}} f(x) = \frac{1}{2}$ and $f\left(\frac{1}{2}\right) = \frac{1}{2}$

Hence, $f(x)$ is continuous at $x = \frac{1}{2}$.

The correct option is (C)

54. We have,

$f(x) = [x] + |1-x|,\ -1 \le x \le 3$

$$= \begin{cases} -x, & -1\le x<0 \\ 1-x, & 0\le x<1 \\ x, & 1\le x<2 \\ 1+x, & 2\le x<3 \\ 5, & x=3 \end{cases}$$

The only doubtful points are $x = -1, 0, 1, 2$ and 3. It can be easily seen that $f(x)$ is differentiable at $x = -1$ but not differentiable at $x = 0, 1, 2$ and 3.

Hence, the required points are 0, 1, 2 and 3.

The correct option is (C)

55. Since $f(x)$ is continuous at $x = 0$,

$\therefore \lim_{x \to 0} f(x) = f(0)$

Take any point $x = a$, then at $x = a$

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} [f(a) + f(h)]$$

$$= f(a) + \lim_{h \to 0} f(h) = f(a) + f(0)$$

$$= f(a+0) = f(a)$$

$\therefore f(x)$ is continuous at $x = a$. Since $x = a$ is any arbitrary point, therefore $f(x)$ is continuous for all x.

The correct option is (C)

56. For $n \in I$,

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [x] \cos \frac{2x-1}{2}\pi$$

$$= n \cos \frac{2n-1}{2}\pi = 0$$

and, $$\lim_{x \to n^-} f(x) = \lim_{x \to n^-} [x] \cos \frac{2x-1}{2}\pi$$

$$= (n-1) \cos \frac{2n-1}{2}\pi = 0$$

Thus, f is continuous for $x = n \in I$. Since the functions $g(x) = [x]$ and $h(x) = \cos \frac{2x-1}{2}\pi$ are continuous on $x \in R - I$, so f is continuous everywhere.

The correct option is (A)

57. Note that $f(x) = 0$ for each integral value of x.

Also, if $0 \le x < 1$, then $0 \le x^2 < 1$

$\therefore [x] = 0$ and $[x^2] = 0 \Rightarrow f(x) = 0$ for $0 \le x < 1$

Next, if $1 \le x < \sqrt{2}$, then

$1 \le x^2 < 2 \Rightarrow [x] = 1$ and $[x^2] = 1$

Thus, $f(x) = [x]^2 - [x^2] = 0$ if $1 \le x < \sqrt{2}$

It follows that $f(x) = 0$ if $0 \le x < \sqrt{2}$

This shows that $f(x)$ must be continuous at $x = 1$.

However, at points x other than integers and not lying between 0 and $\sqrt{2}$, $f(x) \ne 0$

Thus, f is discontinuous at all integers except 1.

The correct option is (D)

58. The function $u = \frac{1}{x-1}$ suffers a discontinuity at the point $x = 1$.

The function $f(x) = \frac{1}{u^2 + u - 2}$ suffers a discontinuity at the points where $u^2 + u - 2 = 0$ i.e., $u = -2$ and $u = 1$. Using these values of u, the corresponding values of x are obtained by solving the equations

$$-2 = \frac{1}{x-1} \text{ and } 1 = \frac{1}{x-1} \text{ i.e., } x = 1/2 \text{ and } x = 2$$

Hence, the composite function is discontinuous at three points $x = 1/2$, $x = 1$ and $x = 2$.

The correct option is (B)

59. Since $|x-1|, |x-1|^2$, etc, are continuous at $x = 1$ $\therefore f(x)$ is continuous at $x = 1$ for all $a_k \in R$.

Also, $|x-1|^2, |x-1|^4$, etc., are all differentiable at $x = 1$, whereas $|x-1|, |x-1|^3$, etc., are not differentiable at $x = 1$. Therefore, $f(x)$ is differentiable at $x = 1$, provided $a_{2k+1} = 0$.

The correct option is (C)

60. $[x+1] = 0$ if $0 \le x + 1 < 1$ i.e., $-1 \le x < 0$

Thus, domain of $f = R \backslash [-1, 0)$

We have, $\sin\left(\frac{\pi}{[x+1]}\right)$ continuous at all points of $R \backslash [-1, 0)$ and $[x]$ continuous on $R \setminus Z$, where Z denotes the set of integers. Thus, the points where f can possibly be discontinuous are ..., $-3, -2, -1, 0, 1, 2, \ldots$

For $0 \le x < 1$, $[x] = 0$ and $\sin\left(\frac{\pi}{[x+1]}\right)$ is defined.

$\therefore f(x) = 0$ for $0 \le x < 1$

Also, f is not defined on $[-1, 0)$, so the continuity of f at 0 means continuity of f from right at 0. Since f is continuous from right at 0, so f is continuous at 0. Hence, the set of points of discontinuity of f is $Z \setminus \{0\}$.

The correct option is (B)

61. We have, $f(0) = 0$, $g(x) = \frac{f(x)}{x}$,

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} f'(x) = f'(0)$$

Thus, the required value is $f'(0)$.

The correct option is (D)

62. We have,

$$\lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{\cos^{-1}(1 - \{h\}^2) \sin^{-1}(1 - \{h\})}{\{h\} - \{h\}^3}$$

$$= \frac{\cos^{-1}(1-h^2) \sin^{-1}(1-h)}{h(1-h^2)}$$

$$= \lim_{h \to 0} \frac{\sin^{-1}(1-h)}{1-h^2} \cdot \lim_{h \to 0} \frac{\cos^{-1}(1-h^2)}{-h}$$

$$= \sin^{-1} 1 \cdot \lim_{\theta \to 0} \frac{\cos^{-1}(1 - 2\sin^2\theta)}{\sqrt{2}\sin\theta}$$

[Putting $1 - h^2 = \cos 2q$]

$$= \frac{\pi}{2\sqrt{2}} \cdot \lim_{\theta \to 0} \frac{2\theta}{\sin\theta} = \frac{\pi}{\sqrt{2}}$$

$$\lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{\cos^{-1}(1 - \{-h\}^2) \sin^{-1}(1 - \{-h\})}{\{-h\} - \{-h\}^3}$$

$$= \lim_{h\to 0} \frac{\cos^{-1}(1-(1-h)^2)\sin^{-1}(1-(1-h))}{(1-h)-(1-h)^3}$$

$$= \lim_{h\to 0} \frac{\cos^{-1}(h(2-h))\sin^{-1}h}{(1-h)(2-h)h}$$

$$= \lim_{h\to 0} \frac{\cos^{-1}(h(2-h))}{(1-h)(2-h)} \cdot \lim_{h\to 0} \frac{\sin^{-1}h}{h}$$

$$= \frac{\cos^{-1}0}{2} \cdot 1 = \pi/4$$

Since R.H.L. ≠ L.H.L., therefore no value of $f(0)$ can make f continuous at $x = 0$

The correct option is (D)

63. We have,

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} [\sin(-h)+\cos(-h)]^{\operatorname{cosec}(-h)}$$

$$= \lim_{h\to 0} (\cos h - \sin h)]^{-\operatorname{cosec} h}$$

$$= \lim_{h\to 0} (1+(\cos h - \sin h - 1))]^{\frac{1}{\cos h - \sin h - 1}\cdot\frac{\cos h - \sin h - 1}{-\sin h}}$$

$$= \left[\lim_{y\to 0} (1+y)^{1/y}\right]^{\lim_{h\to 0}\frac{\cos h - \sin h - 1}{-\sin h}}$$

Now, $\lim_{h\to 0} \frac{\cos h - \sin h - 1}{-\sin h}\left(\frac{0}{0}\right)$

$$= \lim_{h\to 0} \frac{-\sin h - \cos h}{-\cos h} = \frac{0-1}{-1} = 1$$

Thus, $\lim_{x\to 0^-} f(x) = e$

Now, we have,

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} \frac{e^{1/h}+e^{2/h}+e^{3/h}}{ae^{-2+1/-h}+be^{-1+3/h}}$$

$$= \lim_{h\to 0} \frac{e^{-2/h}+e^{-1/h}+1}{(ae^{-2})e^{-2/h}+(be^{-1})}$$

$$= \frac{0+0+1}{(ae^{-2})0+(be^{-1})} = \frac{e}{b}$$

If f is continuous at $x = 0$, then $e = a = \frac{e}{b}$ gives $a = e$ and $b = 1$

The correct option is (A)

64. If f is continuous at $x = 0$, then

$$f(0) = \lim_{x\to 0} \frac{(128a+ax)^{1/8}-2}{(32+bx)^{1/5}-2}$$

As $x \to 0$, the denominator $\to 0$. Thus, for limit to exist the numerator must also $\to 0$. Thus, we have

$$(128a)^{1/8} = 2$$

gives $\quad a = 2$

Now, we have,

$$f(0) = \lim_{x\to 0} \frac{(128a+ax)^{1/8}-2}{(32+bx)^{1/5}-2}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\frac{2}{8}(256+2x)^{-7/8}}{\frac{b}{5}(32+bx)^{-4/5}} = \frac{5}{4b}\cdot\frac{2^{-7}}{2^{-4}} = \frac{5}{32b}$$

gives $\quad b = \frac{5}{32f(0)}$

Hence, we have,

$$\frac{a}{b} = \frac{64}{5}f(0)$$

The correct option is (C)

65. Given that;

$f(xy) = e^{xy-x-y}\{e^y f(x) + e^x f(y)\}\ \forall\, x, y \in R^+$

Putting $x = y = 1$, we get

$f(1) = e^{-1}\{e^1 f(1) + e^1 f(1)\}$

$\Rightarrow\ f(1) = 0$

Now, $f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{f\left\{x\left(1+\frac{h}{x}\right)\right\}-f(x)}{h}$$

$$= \frac{\lim_{h\to 0} e^{x\left(1+\frac{h}{x}\right)-x-\left(1+\frac{h}{x}\right)}\left\{e^{1+\frac{h}{x}}f(x)+e^x f\left(1+\frac{h}{x}\right)\right\}-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{e^h f(x)+e^{h-1-\frac{h}{x}+x} f\left(1+\frac{h}{x}\right)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)(e^h-1)+e^{h-1-\frac{h}{x}+x}\left\{f\left(1+\frac{h}{x}\right)-f(1)\right\}}{h}$$

$(\because f(1) = 0)$

$$= \lim_{h\to 0} \frac{f(x)(e^h-1)}{h} + \frac{\lim_{h\to 0} e^{h-1-\frac{h}{x}+x}\left\{f\left(1+\frac{h}{x}\right)-f(1)\right\}}{\frac{h}{x}\cdot x}$$

$$= f(x) + \frac{e^{x-1}\cdot f'(1)}{x}$$

$$\left\{\because \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)-f(1)}{\frac{h}{x}} = f'(1) \text{ and } \lim_{h\to 0}\frac{e^h-1}{h} = 1\right\}$$

$= f(x) + \dfrac{e^x}{ex} \cdot f'(1)$ $\quad [\because f'(1) = e]$

$\therefore \quad f'(x) = f(x) + \dfrac{e^x}{x}$

$\Rightarrow \quad \dfrac{e^x}{x} = f'(x) - f(x)$

$\Rightarrow \quad \dfrac{1}{x} = \dfrac{e^x f'(x) - f(x)\, e^x}{e^{2x}}$

$\therefore \quad \dfrac{1}{x} = \dfrac{d}{dx}\left\{\dfrac{f(x)}{e^x}\right\}$

Integrating both sides w.r.t. 'x' we get

$\log|x| + c = \dfrac{f(x)}{e^x}$

or, $\quad f(x) = e^x\{\log|x| + c\}$

Since $\quad f(1) = 0 \Rightarrow c = 0$

Thus, $\quad f(x) = e^x \log|x|$

The correct option is (A)

66. Here,

$f(x) = [n + p \sin x]$ is not differentiable at those points where $n + p \sin x$ is integer.

As p is a prime number.

$\Rightarrow n + p \sin x$ is an integer if $\sin x = 1, -1, r/p$

i.e., $x = \dfrac{\pi}{2}, \dfrac{-\pi}{2}, \sin^{-1}\dfrac{r}{p}, \pi - \sin^{-1}\dfrac{r}{p}$, where $0 \le r \le p - 1$

But $x \ne \dfrac{-\pi}{2}, 0.$

$\therefore$ Function is not differentiable at $x = \dfrac{\pi}{2}, \sin^{-1}\dfrac{r}{p},$

$\pi - \sin^{-1}\dfrac{r}{p}$, where $0 < r \le p - 1$

So, the required number of points are,

$= 1 + 2(p - 1) = 2p - 1$

The correct option is (C)

67. Here, $x = 2t - |t - 1|$ and $y = 2t^2 + t|t|$

Now, when $t < 0$;

$x = 2t - \{-(t - 1)\} = 3t - 1$

and $y = 2t^2 - t^2 = t^2 \Rightarrow y = \dfrac{1}{9}(x+1)^2$

when $\quad 0 \le t < 1$;

$x = 2t - (-(t - 1)) = 3t - 1$

and $y = 2t^2 + t^2 = 3t^2 \Rightarrow y = \dfrac{1}{3}(x-1)^2$

when $t > 1$;

$x = 2t - (t - 1) = t + 1$

and $y = 2t^2 + t^2 = 3t^2 \Rightarrow y = 3(x - 1)^2$

Thus,

$$y = f(x) = \begin{cases} \dfrac{1}{9}(x+1)^2, & x < -1 \\ \dfrac{1}{3}(x+1)^2, & -1 \le x < 2 \\ 3(x-1)^2, & x \ge 1 \end{cases}$$

We have to check continuity and differentiability at $x = -1$ and 2.

Differentiability at $x = -1$;

$$\text{L.H.D.} = Lf'(-1) = \lim_{h\to 0}\frac{f(-1-h) - f(-1)}{-h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{9}(-1-h+1)^2 - 0}{-h} = 0$$

$$\text{R.H.D.} = Rf'(-1) = \lim_{h\to 0}\frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{3}(-1+h+1)^2 - 0}{h} = 0$$

Therefore, $f(x)$ is differentiable and hence continuous at $x = -1$

Differentiability at $x = 2$;

$$\text{L.H.D. } Lf'(2) = \lim_{h\to 0}\frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{3}(2-h+1)^2 - 3}{-h} = 2$$

$$\text{R.H.D. } Rf'(2) = \lim_{h\to 0}\frac{f(2+f) - f(2)}{h}$$

$$= \lim_{h\to 0}\frac{3(2+h-1)^2 - 3}{h} = 6$$

Hence, $f(x)$ is not differentiable at $x = 2$. Since $Lf'(2)$ and $Rf'(2)$ are finite, therefore $f(x)$ is continuous at $x = 2$. Hence, $f(x)$ is continuous for all x and differentiable for all x except $x = 2$.

The correct option is (B)

68. As $f(x)$ is continuous for all $x \in R$

Thus,

$$\lim_{x\to\sqrt{3}} f(x) = f(\sqrt{3})$$

where $\quad f(x) = \dfrac{x^2 - 2x + 2\sqrt{3} - 3}{\sqrt{3} - x}, x \ne \sqrt{3}$

$$\therefore \quad \lim_{x\to\sqrt{3}} f(x) = \lim_{x\to\sqrt{3}}\frac{x^2 - 2x + 2\sqrt{3} - 3}{\sqrt{3} - x}$$

$$= \lim_{x\to\sqrt{3}} \frac{(2-\sqrt{3}-x)(\sqrt{3}-x)}{(\sqrt{3}-x)}$$

$$= 2(1-\sqrt{3})$$

Thus, $f(\sqrt{3}) = 2(1-\sqrt{3})$

The correct option is (C)

69. We know that $\lim_{n\to\infty} x^{2n} = \begin{cases} 0 & \text{if } x^2 < 1 \\ \infty & \text{if } x^2 > 1 \end{cases}$

$\therefore$ for $x^2 < 1$, we have

$$f(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}}$$

$$= \log(2+x)$$

and for $x^2 > 1$,

$$f(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}}$$

$$= \lim_{n\to\infty} \frac{\frac{\log(2+x)}{x^{2n}} - \sin x}{\frac{1}{x^{2n}}+1} = -\sin x$$

$\therefore \lim_{x\to 1^-} f(x) = \log(2+1) = \log 3$ and $\lim_{x\to 1^+} f(x) = -\sin 1$

$\therefore$ jump of discontinuity at $x = 1$

$$= \left|\lim_{x\to 1^+} f(x) - \lim_{x\to 1^-} f(x)\right| = \sin 1 + \log 3$$

The correct option is (B)

70. $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} \frac{h}{e^{1/h}+1} = 0 \left(\because \lim_{h\to 0} e^{1/h} = \infty\right)$

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} \frac{-h}{e^{-1/h}+1} = 0 \left[\because \lim_{h\to 0} e^{-1/h} = 0\right]$

Therefore, $f(x)$ is continuous at $x = 1$

The correct option is (A)

71. Since $|f(x) - f(y)| \le |x-y|^2$ $x \ne y$

$$\therefore \left|\frac{f(x)-f(y)}{x-y}\right| \le |x-y|$$

Taking limit as $y \to x$, we get

$$\lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{y\to x}|x-y|$$

$$\Rightarrow \left|\lim_{y\to x}\frac{f(x)-f(y)}{x-y}\right| \le \left|\lim_{y\to x}(x-y)\right|$$

$\Rightarrow |f'(x)| \le 0 \Rightarrow |f'(x)| = 0$ $[\because |f'(x)| \ge 0]$

$\therefore f'(x) = 0 \Rightarrow f(x) = 6$ (constant)

$\therefore h(x) = \int f(x)\,dx = \int c\,dx = c\,x + d$, where d is constant of integration

$\therefore h(x)$ of a linear function of x which is continuous for all $x \in R$.

The correct option is (A)

72. If $f(a) = 0$, then obviously, $x = a$ is the solution

Let $f(a) > a$. Since $g(x) = f(x) - x$, therefore $g(a) > 0$ and $g(f(a)) = f(f(a)) - f(a) = a - f(a) < 0$

Since g is continuous from R to R so at least for one $c \in (a, f(a))$, $g(c) = 0$

Similarly, we can argue for $f(a) < a$

The correct option is (D)

73. As f is continuous on R, so $f(0) = \lim_{x\to 0} f(x)$

$= \lim_{n\to\infty} f(x_n)$ for any sequence x_n such that $\lim_{n\to\infty} x_n = 0$.

Thus, $f(0) = \lim_{n\to\infty} f(1/4n)$

$$= \lim_{n\to\infty}\left((\sin e^n)e^{-n^2} + \frac{1}{1+1/n^2}\right)$$

$$= 0 + 1 = 1$$

$\left(\lim_{n\to\infty}(\sin e^n)e^{-n^2} = 0 \text{ as } \left|(\sin e^n)e^{-n^2}\right| \le e^{-n^2}\right.$ and $\left.e^{-n^2} \to 0 \text{ as } n \to \infty\right)$

The correct option is (A)

74. We have $f'(x) + f^2(x) \ge -1$

$$\Rightarrow \frac{f'(x)}{1+f^2(x)} + 1 \ge 0, \text{ for } x \in (a, b) \qquad (1)$$

$$\therefore \frac{d}{dx}(\tan^{-1} f(x) + x) = \frac{f'(x)}{1+(f(x))^2} + 1 \ge 0 \text{ (from (1))}$$

$\Rightarrow h(x) = \tan^{-1} f(x) + x$, is a non-decreasing function in the interval (a, b)

$$\Rightarrow \lim_{x\to a} h(x) \le \lim_{x\to b} h(x)$$

$$\Rightarrow \lim_{x\to a^+}(\tan^{-1} f(x) + x) \le \lim_{x\to b^-}(\tan^{-1} f(x) + x)$$

$$\Rightarrow \frac{\pi}{2} + a \le -\frac{\pi}{2} + b$$

Hence, $b - a$ ³π

The correct option is (B)

75. We have,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)+f(h)+2hx-1-1f(x)}{h}$$

$$= \lim_{h\to 0}\left\{2x + \frac{f(h)-1}{h}\right\}$$

Now, substituting $x = y = 0$ in the given functional relation, we get,

$$f(0) = f(0) + f(0) + 0 - 1 \Rightarrow f(0) = 1$$

$$\therefore \quad f'(x) = 2x + \lim_{h\to 0} \frac{f(h) - f(0)}{h} = 2x + f'(0)$$

$$\Rightarrow \quad f'(x) = 2x + \cos\alpha$$

Integrating, $f(x) = x^2 + x\cos\alpha + C$

Here, $x = 0$ and $f(0) = 1$

$\therefore \quad 1 = C$

$$\Rightarrow \quad f(x) = x^2 + x\cos\alpha + 1$$

It is a quadratic in x with discriminant

$D = \cos^2\alpha - 4 < 0$

and coefficient of $x^2 = 1 > 0$

$\therefore \quad f(x) > 0 \ \forall x \in R$

The correct option is (C)

76. Since $f(x)$ is differentiable at $x = 0$

$$\Rightarrow \quad \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = a \text{ (say)} \quad (1)$$

$$\text{Now, } f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f\left(\frac{3x+3h}{3}\right) - f\left(\frac{3x+3\cdot 0}{3}\right)}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(3x) + f(3h) + f(0) - f(3x) - f(0) - f(0)}{3h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(3h) - f(0)}{3h}$$

$$\Rightarrow f'(x) = f'(0)$$

$\Rightarrow f'(x) = a$ [from (1)] (say)

$\therefore \quad f(x) = ax + b$, which is linear

The correct option is (A)

More than One Option Correct Type

77. Continuity at $x = 0$

$$\text{L.H.L.} = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} (-h)^p \cos\frac{1}{h}$$

$= 0$ if $p > 0$

$$\text{R.H.L.} = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} h^p \cos\frac{1}{h}$$

$= 0$ if $p > 0$

and, $f(0) = 0$.

$\therefore$ ***f*(*x*) is continuous at *x* = 0 if *p* > 0**

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{(-h)^p \cos\frac{1}{h} - 0}{-h}$$

$$= \lim_{h\to 0} (-h)^{p-1} \cos\frac{1}{h} = 0 \text{ if } p > 0,$$

$$\text{and, } Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^p \cos\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0} h^{p-1} \cos\frac{1}{h} = 0 \text{ if } p > 1$$

$\therefore$ ***f*(*x*) is differentiable at *x* = 0 if *p* > 1.**

The correct option is (A, B)

78 We have, $g(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

For $x \neq 0$,

$$g'(x) = x^2 \cos\frac{1}{x}\left(-\frac{1}{x^2}\right) + 2x\sin\frac{1}{x}$$

$$= -\cos\frac{1}{x} + 2x\sin\frac{1}{x}$$

For $x = 0$

$$g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \frac{x^2\sin\frac{1}{x} - 0}{x}$$

$$= \lim_{x\to 0} x\sin\frac{1}{x} = 0$$

$$\therefore \quad g'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

***g*′ is not continuous at *x* = 0** as $\cos\frac{1}{x}$ is not continuous at $x = 0$. Also, f is not differentiable at $x = 0$.

The correct option is (A, B)

79. $f(x) = \begin{cases} 1 - x, & x \leq -1 \\ 2, & -1 < x \leq 1 \\ 1 + x, & x > 1 \end{cases}$

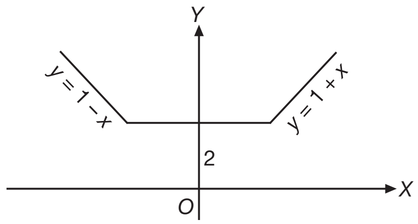
$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (1-x) = 2 = \lim_{x\to -1^+} f(x)$$

and, $\lim_{x \to 1} f(x) = 2$, so f is continuous at all points.

$$f'(-1^-) = \lim_{h \to o^-} \frac{f(-1-h) - f(-1)}{-h}$$

$$= \lim_{h \to o^-} \frac{1+1+h-2}{-h} = -1$$

$f'(-1^+) = 0$. Similarly, $f'(1^-) = 0$ and $f'(1+) = 1$, so ***f* is differentiable everywhere except at *x* = – 1, 1.**



The correct option is (A, C)

80. Let $x = n, n \in Z$

Then, L.H.L. = $\lim_{\substack{x \to n \\ x < n}} (x) = n$; R.H.L. = $\lim_{\substack{x \to n \\ x < n}} (x) = n + 1$

Since, L.H.L. ≠ R.H.L., therefore $f(x)$ is discontinuous at all integers n.

Now, let $x = p, n < p \le n + 1$, where n is an integer.

Then, L.H.L. = $\lim_{\substack{x \to p \\ x < p}} (x) = n + 1$,

R.H.L. = $\lim_{\substack{x \to p \\ x > p}} (x) = n + 1$

$f(p) = (p) = n + 1$

Since L.H.L. = R.H.L. = $f(p)$, therefore, ***f*(*x*) is continuous at all non-integral points** p.

The correct option is (B, C)

81. The given function is clearly continuous at all points except possibly at $x = \pm 1$.

For $f(x)$ to be continuous at $x = 1$, we must have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{x \to 1} ax^2 + b = \lim_{x \to 1} \frac{1}{|x|}$$

$$\Rightarrow \quad a + b = 1 \qquad (1)$$

Now, for $f(x)$ to be differentiable at $x = 1$, we must have

$$\lim_{x \to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1}$$

$$\Rightarrow \lim_{x \to 1} \frac{ax^2 + b - 1}{x - 1} = \lim_{x \to 1} \frac{\frac{1}{|x|} - 1}{x - 1}$$

$$\Rightarrow \lim_{x \to 1} \frac{ax^2 - a}{x - 1} = \lim_{x \to 1} \frac{\frac{1}{x} - 1}{x - 1}$$

$\{\because a + b = 1 \therefore b - 1 = -a\}$

$$\Rightarrow \lim_{x \to 1} a(x+1) = \lim_{x \to 1} \frac{-1}{x} \Rightarrow a = -\frac{1}{2}$$

Putting $a = -\frac{1}{2}$ in (1), we get $b = \frac{3}{2}$

The correct option is (B, C)

82. We have, $\lim_{h \to 0} f(0 - h) = \lim_{h \to 0} [-h \sin(-ph)]$

$$= \lim_{h \to 0} [h \sin \pi h] = 0,$$

$\lim_{h \to 0} f(0 + h) = \lim_{h \to 0} [h \sin ph] = 0$. and $f(0) = 0$

∴ *f*(*x*) is continuous at *x* = 0.

It can be easily seen that ***f*(*x*) is continuous in (– 1, 0).** $f(x)$ is not differentiable at $x = 1$ but **it is differentiable in (– 1, 1).**

The correct option is (A, B, D)

83. We have,

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \left[\frac{-h(3e^{-1/h} + 4)}{2 - e^{-1/h}} - 0 \right] \cdot \left(\frac{-1}{h} \right)$$

$$= \frac{0 + 4}{2 - 0} = 2$$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \left[\frac{h(3e^{1/h} + 4)}{2 - e^{1/h}} - 0 \right] \frac{1}{h}$$

$$= \lim_{h \to 0} \left(\frac{3 + 4e^{-1/h}}{2e^{-1/h} - 1} \right) = \frac{3 + 0}{0 - 1} = -3$$

Since $Lf'(0) \ne Rf'(0)$, ∴ ***f*(*x*) is not differentiable at *x* = 0.** But ***f*(*x*) is continuous at *x* = 0** (as $Lf'(0)$ and $Rf'(0)$ are finite).

The correct option is (A, D)

84. Clearly, $x = 1$ is a point of discontinuity of the function $f(x) = \frac{1}{1 - x}$.

If $x \ne 1$, then $(fof)(x) = f[f(x)] = f\left(\frac{1}{1-x}\right) = \frac{x-1}{x}$, which is discontinuous at $x = 0$.

If $x \ne 0$ and $x \ne 1$, then

$$(fofof)(x) = f[(fof)(x)] = f\left(\frac{x-1}{x}\right) = x,$$

which is continuous everywhere.

Hence, $f^{3n}(x) = (fofof)^n(x) = x$, which is continuous everywhere.

So, the only points of discontinuity are $x = 0$ and $x = 1$.

The correct option is (B, C)

85. For $x \neq -1$, we have

$$f(x) = 1 - 2x + 3x^2 - 4x^3 + \ldots\infty$$
$$= (1+x)^{-1} = \frac{1}{1+x}$$

$$\lim_{h\to 0} f(-1-h) = \lim_{h\to 0} \frac{1}{1-1-h} \to -\infty$$

So, $f(x)$ is not continuous at $x = -1$

Also, $$\lim_{h\to 0} \frac{f(-1-h) - f(-1)}{(-1-h)-(-1)} = \lim_{h\to 0} \frac{\frac{-1}{h} - 1}{-h}$$

$$= \lim_{h\to 0} \frac{1+h}{h^2} \to \infty$$

So, $f(x)$ is not derivable at $x = -1$.

Hence, **$f(x)$ is neither continuous nor derivable at $x = -1$.**

The correct option is (B, D)

86. We have,

$$F(x) = f(x), |x| < 1 \qquad \left[\lim_{n\to\infty} x^{2n} = 0 \text{ for } |x| < 1\right]$$

$$= \frac{f(x) + g(x)}{2}, \ |x| = 1$$

$$= \lim_{n\to\infty} \frac{x^{-2n} f(x) + g(x)}{x^{-2n} + 1}$$

$$= g(x), |x| > 1 \qquad \left[\lim_{n\to\infty} x^{-2n} = 0 \text{ for } |x| > 1\right]$$

Thus, the function $F(x)$ shall be continuous everywhere if $f(x)$ and $g(x)$ are continuous everywhere and if $F(x)$ is continuous at $x = \pm 1$, we have

$$f(1) = \frac{f(1) + g(1)}{2} = g(1) \Rightarrow f(1) = g(1)$$

and, $$f(-1) = \frac{f(-1) + g(-1)}{2} = g(-1) \Rightarrow f(-1) = g(-1)$$

The correct option is (A, B)

87. We have,

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} \left(1 + \frac{ah + bh^3}{h^2}\right)^{1/h}$$

$$= \lim_{h\to 0} e^{\frac{1}{h} ln\left(1 + \frac{ah+bh^3}{h^2}\right)}$$

For limit to exist, we have

$$\lim_{h\to 0} \frac{ah + bh^3}{h^2} = 0 \text{ i.e., } \quad \lim_{h\to 0} \frac{a + bh^2}{h} = 0,$$

which is possible only if $a = 0$.

Now, we have

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} (1 + bh)^{1/h}$$

$$= \lim_{h\to 0} (1 + bh)^{b/bh} = e^b$$

For $f(x)$ to be continuous at $x = 0$, we have

$$\lim_{x\to 0^+} f(x) = f(0) \Rightarrow e^b = 3$$

$\therefore \qquad b = ln\ 3$

The correct option is (A, D)

88. We have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin 3x + a \sin 2x + b \sin x}{x^5} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{3\cos 3x + 2a\cos 2x + b\cos x}{5x^4}$$

For a finite limit to exist, the numerator must be 0 at $x = 0$, since the denominator is 0 at $x = 0$

i.e., $\qquad 3 + 2a + b = 0 \qquad (1)$

Now, we have,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{3\cos 3x + 2a\cos 2x + b\cos x}{5x^4} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-9\sin 3x - 4a\sin 2x - b\sin x}{20x^3} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-27\cos 3x - 8a\cos 2x - b\cos x}{60x^2}$$

For a finite limit to exist, the numerator must be 0 at $x = 0$, since the denominator is 0 at $x = 0$

i.e. $\quad 27 + 8a + b = 0 \qquad (2)$

Now, we have,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{81\sin 3x + 16a\sin 2x + b\sin x}{120x}$$

$$= \lim_{x\to 0} \frac{81}{40}\left(\frac{\sin 3x}{3x}\right) + \frac{16a}{60}\left(\frac{\sin 2x}{2x}\right) + \frac{b}{120}\left(\frac{\sin x}{x}\right)$$

$$= \frac{81}{40} + \frac{16a}{60} + \frac{b}{120}$$

Solving equations (1) and (2), we have

$a = -4, b = 5$

For f to be continuous at $x = 0$, we have

$$f(0) = \lim_{x\to 0} f(x) = \frac{81}{40} - \frac{64}{60} + \frac{5}{120} = 1$$

$f(0) = 1$

The correct option is (A, B, D)

89. We have,

$$\lim_{x\to 0} \frac{2f(x) - 3af(2x) + bf(8x)}{\sin^2 x}$$

$$= \lim_{x\to 0} \frac{2f(x) - 3a\, f(2x) + bf(8x)}{x^2}$$

For the limit to exist, we have

$2f(0) - 3af(0) + b\,f(0) = 0$

i.e., $3a - b = 2 \qquad [\because f(0) \neq 0, \text{ given}] \quad (1)$

$$= \lim_{x\to 0} \frac{2f'(x) - 6a\, f'(2x) + 8b\, f'(8x)}{2x}$$

For the limit to exist, we have

$2f'(0) - 6af'(0) + 8bf'(0) = 0$

i.e., $3a - 4b = 1$ $\quad [\because f'(0) \neq 0$, given$] \quad (2)$

Solving equations (1) and (2), we have

$a = 7/9$ and $b = 1/3$

The correct option is (B, C)

90. We have, $f|x| = \begin{cases} |x| - 3 & , |x| < 0 \\ |x|^2 - 3|x| + 2, & |x| \geq 0 \end{cases}$

Since $|x| < 0$ is not possible, so we get,

$f(|x|) = |x|^2 - 3|x| + 2, |x| \geq 0$

$$= \begin{cases} x^2 + 3x + 2, & x < 0 \\ x^2 - 3x + 2, & x \geq 0 \end{cases} \quad (1)$$

Again,

$$|f(x)| = \begin{cases} |x - 3| & x < 0 \\ |x^2 - 3x + 2| & x \geq 0 \end{cases}$$

$$= \begin{cases} (x^2 - 3x + 2), & 0 \leq x < 1 \\ -(x^2 - 3x + 2), & 1 \leq x < 2 \\ (x^2 - 3x + 2), & 2 \leq x \end{cases} \quad (2)$$

From (1) and (2), we get

$g(x) = f(|x|) + |f(x)|$

$$= \begin{cases} x^2 + 2x + 5, & x < 0 \\ 2x^2 - 6x + 4, & 0 \leq x < 1 \\ 0 \quad , & 1 \leq x < 2 \\ 2x^2 - 6x + 4, & x \geq 2 \end{cases}$$

and, $g'(x) = \begin{cases} 2x + 2, & x < 0 \\ 4x - 6, & 0 < x < 1 \\ 0 \quad , & 1 < x < 2 \\ 4x - 6, & x > 2 \end{cases}$

Clearly, **$g(x)$ is continuous in $R - \{0\}$ and differentiable in $R - \{0, 1, 2\}$**

The correct option is (A, C)

91. Here, $f(x) = x^3 - x^2 + x + 1$

$\Rightarrow f'(x) = 3x^2 - 2x + 1$, which is strictly increasing in (0, 2)

$\therefore \quad g(x) = \begin{cases} f(x); & 0 \leq x \leq 1 \\ 3 - x; & 1 < x \leq 2 \end{cases}$

[as $f(x)$ is increasing so $f(x)$ is maximum when $0 \leq t \leq x$]

So, $g(x) = \begin{cases} x^3 - x^2 + x + 1; & 0 \leq x \leq 1 \\ 3 - x & ; 1 < x \leq 2 \end{cases}$

Also, $g'(x) = \begin{cases} 3x^2 - 2x + 1; & 0 \leq x \leq 1 \\ -1 & ; 1 < x \leq 2 \end{cases}$

which clearly shows **$g(x)$ is continuous for all $x \in [0, 2]$ but $g(x)$ is not differentiable at $x = 1$**

The correct option is (A, D)

92. Here, $f(x) = x^4 - 8x^3 + 22x^2 - 24x$

$\Rightarrow \quad f'(x) = 4x^3 - 24x^2 + 44x - 24$

or, $\quad f'(x) = 4(x - 1)(x - 2)(x - 3)$

which shows $f(x)$ is increasing in $[1, 2] \cup [3, \infty)$ and decreasing in $(-\infty, 1] \cup [2, 3]$.

Thus, minimum $f(x)$;$x \leq t \leq x + 1, -1 \leq x \leq 1$

$\Rightarrow$ minimum $f(x) = \begin{cases} f(x+1), & -1 \leq x \leq 0 \\ f(1) \quad , & 0 < x \leq 1 \end{cases}$

Thus, $g(x) = \begin{cases} f(x+1), & -1 \leq x \leq 0 \\ f(1), & 0 < x \leq 1 \\ x - 10, & x > 1 \end{cases}$

$$= \begin{cases} (x+1)^4 - 8(x+1)^3 + 2^2(x+1)^2 - 24, & -1 \leq x \leq 0 \\ 1 - 8 + 22 - 24, & 0 < x \leq 1 \\ x - 10, & x > 1 \end{cases}$$

$$\Rightarrow g(x) = \begin{cases} x^4 - 4x^3 + 4x^2 - 9, & -1 \leq x \leq 0 \\ -9, & 0 < x \leq 1 \\ x - 10, & x > 1 \end{cases}$$

Also, $g'(x) = \begin{cases} 4x^3 - 12x^2 + 8x, & -1 \leq x \leq 0 \\ 0, & 0 < x \leq 1 \\ 1, & x > 1 \end{cases}$

which clearly shows **$g(x)$ is continuous in $[-1, \infty)$ but not differentiable at $x = 1$**

The correct option is (A, D)

93. $f(x) = [\tan x] + \sqrt{\tan x - [\tan x]} = [t] + \sqrt{t - [t]}$, where $t = \tan x$. Clearly, $0 \leq t < \infty$ at $0 \leq x < \pi/2$. Possible points of discontinuity may be, at which $t \in N$.

Let $t = k \in N$

L.H.L. at $t = k = \lim_{t\to k^-} [t] + \sqrt{t - [t]}$

$$= \lim_{h\to 0} [k - h] + \sqrt{(k - h) - [k - h]}$$

$$= \lim_{h\to 0} \left\{k - 1 + \sqrt{k - h - k + 1]}\right\} = k$$

R.H.L. at $t = k = \lim_{t\to k+} [t] + \sqrt{t - [t]}$

$$= \lim_{h\to 0} [k + h] + \sqrt{k + h - [k + h]}$$

$= \lim_{h\to 0} k + \sqrt{k+h-k} = k$

$\therefore$ The function is continuous at $t = k \in N$

Thus, function **$f(x)$ is continuous for all $x \in$** $[o, \pi/2)$

The correct option is (A, C)

94. $\lim_{x\to 0^+} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} = \lim_{x\to 0^+} \frac{1 - e^{-2/x}}{1 + e^{-2/x}} = 1$

and, $\lim_{x\to 0^-} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} = \lim_{x\to 0^-} \frac{e^{2/x} - 1}{e^{2/x} + 1} = -1$

Hence, $\lim_{x\to 0^-} f(x)$ exists if $g'(0) = 0$

If $g(x) = ax + b$, $a \neq 0$ then $\lim_{x\to 0^+} f(x) = a$ and

$\lim_{x\to 0^-} f(x) = -a$. Hence $\lim_{x\to 0} f(x)$ exists if $g(x) = x^2$

or, $g(x) = x^3 h(x)$, where $h(x)$ is a polynomial.

The correct option is (C, D)

95. Since, $f(x)$ is continuous at $x = 0$ so at $x = 0$ both left and right limits must exist and both must be equal to 3.

Now,

$$\frac{a(1 - x\sin x) + b\cos x + 5}{x^2}$$

$$= \frac{(a+b+5) + \left(-a - \frac{b}{2}\right)x^2 + \ldots}{x^2} = 3$$

[By expansion of sin x and cos x]

If $\lim_{x\to 0^-} f(x) = 3$ exists, then $a + b + 5 = 0$

and, $-a - \frac{b}{2} = 3 \Rightarrow a = -1$ and $b = -4$

Since, $\lim_{x\to 0^+} \left(1 + \left(\frac{cx + dx^3}{x^2}\right)\right)^{1/x}$ exists.

$$\Rightarrow \lim_{x\to 0^+} \frac{cx + dx^3}{x^2} = 0 \Rightarrow c = 0$$

Now, $\lim_{x\to 0^+} (1 + dx)^{1/x} = \lim_{x\to 0^+} \left[(1 + dx)^{1/dx}\right]^d = e^d$

So, $e^d = 3 \Rightarrow d = \log_e 3$

Hence, $a = -1$, $b = -4$, $c = 0$ and $d = \log_e 3$

The correct option is (A, B, C, D)

Passage Based Questions

96. We have,

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{3-1}{-h} \to -\infty$$

$\therefore$ **$f(x)$ is not differentiable at $x = 0$**

Also, if $x < 0$ or $x \geq 0$ then $|x| \geq 0$

$\therefore$ $f(|x|) = 2|x| + 1$ for all x.

$$\therefore \quad Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{2h + 1 - 1}{h} = 2$$

$$\text{and, } Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{2(-h) + 1 - 1}{-h} = 2$$

$\therefore$ **$f(|x|)$ is differentiable at** $x = 0$.

The correct option is (C)

97. $L(gof)'\left(\frac{\pi}{2}\right)$

$$= \lim_{h\to 0} \frac{(gof)\left(\frac{\pi}{2} - h\right) - (gof)\left(\frac{\pi}{2}\right)}{-h}$$

$$= \lim_{h\to 0} \frac{\left[\cos\left(\frac{\pi}{2} - h\right) + 2\right] - \left[\cos\frac{\pi}{2} + 2\right]}{-h}$$

$$= \lim_{h\to 0} \frac{[\sin h + 2] - [2]}{-h} = \lim_{h\to 0} \frac{2-2}{-h} = 0$$

$$R(fog)'\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{(gof)\left(\frac{\pi}{2} + h\right) - (gof)\left(\frac{\pi}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left[\cos\left(\frac{\pi}{2} + h\right) + 2\right] - \left[\cos\frac{\pi}{2} + 2\right]}{h}$$

$$= \lim_{h\to 0} \frac{[-\sin h + 2] - [2]}{h}$$

$$= \lim_{h\to 0} \frac{1-2}{h} \to -\infty$$

$\therefore$ (gof) is not differentiable at $x = \pi/2$.

The correct option is (D)

98. $f'(k-0) = \lim_{h\to 0} \frac{[k-h]\sin\pi(k-h) - [k]\sin\pi k}{-h}$

$$= \lim_{h\to 0} \frac{(-1)^{k-1}(k-1)\sin\pi h - k \times 0}{-h}$$

$$= \lim_{h \to 0} \frac{(-1)^{k-1} \cdot (k-1)\sin \pi h}{-h}$$

$= (-1)^k \cdot (k-1)\, \pi.$

The correct option is (A)

99. Differentiability at $x = 0$:

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-h\,(e^{-1/h} - e^{1/h})}{-h\,(e^{-1/h} + e^{1/h})} = \lim_{h \to 0} \frac{e^{-2/h} - 1}{e^{-2/h} + 1} = -1.$$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h\,(e^{1/h} - e^{-1/h})}{h\,(e^{1/h} + e^{-1/h})}$$

$$= \lim_{h \to 0} \frac{1 - e^{-2/h}}{1 + e^{-2/h}} = 1$$

Since $Lf'(0) \neq Rf'(0)$, $\therefore f(x)$ is not differentiable at $x = 0$.

But since $Lf'(0)$ and $Rf'(0)$ are finite, therefore $f(x)$ is continuous at $x = 0$.

Hence, **$f(x)$ is continuous everywhere but not differentiable at $x = 0$.**

The correct option is (A)

100. We have,

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} (\log a)^n = \sum_{n=0}^{\infty} \frac{(x \log a)^n}{n!}$$

$$= e^{x \log a} = e^{\log a^x} = a^x$$

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{a^{-h} - 1}{-h}$$

$= \log_e a$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{a^h - 1}{h}$$

$= \log_e a$

Since $Lf'(0) = Rf'(0)$,

$\therefore f(x)$ is differentiable at $x = 0$.

Since every differentiable function is continuous, therefore, $f(x)$ is continuous at $x = 0$.

The correct option is (B, D)

101. We have, for $x > 2$

$$f(x) = \int_0^5 \{5 + |1 - t|\}\, dt$$

$$= \int_0^1 \{5 + 1 - t\}\, dt + \int_1^x \{5 + t - 1\}\, dt$$

[Since $x > 2$]

$$= \left(6t - \frac{t^2}{2}\right)_0^1 + \left(4t + \frac{t^2}{2}\right)_1^x$$

$$= 6 - \frac{1}{2} + 4x + \frac{x^2}{2} - 4 - \frac{1}{2}$$

$$= 1 + 4x + \frac{x^2}{2}$$

$$f(x) = \begin{cases} 1 + 4x + x^2, & \text{if } x > 2 \\ 5x + 1, & \text{if } x \leq 2 \end{cases}$$

We have, $$Rf'(2) = \lim_{h \to 0} \frac{f(2+h) - h(2)}{h}$$

$$= \lim_{h \to 0} \frac{1 + 4(2+h) + \frac{(2+h)^2}{2} - 11}{h}$$

$$= \lim_{h \to 0} \frac{11 + 6h + \frac{h^2}{2} - 11}{h} = 6$$

and, $$Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{5(2-h) - 1 - 11}{h}$$

$$= \lim_{h \to 0} \frac{11 - 5h - 11}{h} = -5$$

$\therefore f(x)$ is not differentiable at $x = 2$

Since $Rf'(2)$ and $Lf'(2)$ are finite, therefore $f(x)$ is continuous at $x = 2$

The correct option is (A, D)

102. We have,

$$\lim_{h \to 0} f(5-h) = \lim_{h \to 0} \tan^{-1} \frac{1}{(5-h) - 5}$$

$$= \lim_{h \to 0} \tan^{-1}\left(\frac{-1}{h}\right)$$

$$= \tan^{-1}(-\infty) = \frac{-\pi}{2}$$

and $$\lim_{h \to 0} f(5+h) = \lim_{h \to 0} \tan^{-1} \frac{1}{(5+h) - 5}$$

$$= \lim_{h \to 0} \tan^{-1}\left(\frac{1}{h}\right)$$

$$= \tan^{-1}(\infty) = \frac{\pi}{2}.$$

Since $\lim_{h \to 0} f(5-h) \neq \lim_{h \to 0} f(5+h)$, therefore, $f(x)$ has **discontinuity of the first kind at $x = 5$.**

The correct option is (A)

103. The function $u = \tan x$ is discontinuous at $n\pi \pm \pi/2,\ n \in I$

The function $f(x) = \dfrac{1 - u^2}{2 + u^2}$

is continuous at every $u \in R$. Hence, $f(x)$ is continuous on $x \in R \sim n \in I$

Now, we have,

$$\lim_{x \to n\pi \pm \frac{\pi}{2}} f(x) = \lim_{u \to \infty} \frac{1-u^2}{2+u^2} = \lim_{u \to \infty} \frac{\frac{1}{u^2}-1}{\frac{2}{u^2}+1} = -1$$

Hence, **the points $x = n\pi \pm \frac{\pi}{2}$ $n \in I$ have removable discontinuity.**

The correct option is (C)

104. The function

$f(x) = t^3 = (x-1)^3, x \le 0$
$= (x+1)^3, 0 < x < 1$
$= 1, x = 1$
$= (3-x)^3, 1 < x$

may have discontinuities at $x = 0, 1$.

At $x = 0$, we have

$$\lim_{x \to 0^-} f(x) = f(0) = (0-1)^3 = -1$$

$$\lim_{x \to 0^+} f(x) = (0+1)^3 = 1$$

Since $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$ both exist but are not equal, therefore f **has discontinuity of first kind at $x = 0$.**

At $x = 1$, we have $\lim_{x \to 1^-} f(x) = (1+1)^3 = 8$ and $f(1) = 1$.

Since $\lim_{x \to 1} f(x) = 8 \ne f(1)$, therefore, **$f$ has a removable discontinuity at $x = 1$.**

The correct option is (A, B)

105. We have,

$[\cos x] = 1, x = 0$
$= 0, 0 < x \le \pi/2$
$= -1, \pi/2 < x < 3\pi/2$
$= 0, \quad 3\pi/2 \le x < 2\pi$
$= 1, x = 2\pi$

Therefore, $f(x) = \dfrac{1}{[\cos x]} = 1, x = 0$

$\infty, 0 < x \le \pi/2$
$-1, \pi/2 < x < 3\pi/2$
$\infty, 3\pi/2 \le x < 2\pi$
$1, x = 2\pi$

Clearly, $f(x)$ has discontinuity of second kind from left at $x = \pi/2$

The correct option is (C)

Match the Column Type

106. I. We have, $fog = I$

$\Rightarrow \quad (fog)(x) = x$ for all x

$\Rightarrow \quad f[g(x)] = x \Rightarrow f'[g(x)] \cdot g'(x) = 1$

$\Rightarrow \quad f'[g(a)] \cdot g'(a) = 1 \Rightarrow f'[g(a)] = \dfrac{1}{g'(a)} = \dfrac{1}{2}$ $\quad [g'(a) = 2]$

$\Rightarrow f'(b) = \dfrac{1}{2}$ $\quad [\because g(a) = b]$.

The correct option is (C)

II. We have,

$$\lim_{h \to 0} f(0-h) = \lim_{h \to 0} [1 + |\sin(-h)|]^{a/|\sin(-h)|}$$

$$= \lim_{h \to 0} (1 + \sin h)^{a/\sin h}$$

$$= \lim_{h \to 0} [(1 + \sin h)^{1/\sin h}]^a = e^a,$$

$$\lim_{h \to 0} f(0+h) = \lim_{h \to 0} e^{\tan 2h/\tan 3h}$$

$$= e^{\lim_{h \to 0}\left[\frac{\tan 2h}{2h} \times \frac{2}{3} \times \frac{3h}{\tan 3h}\right]} = e^{1 \times \frac{2}{3} \times 1} = e^{\frac{2}{3}}$$

and $\quad f(0) = b$

For f to be continuous at $x = 0$, we must have

$$\lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(0+h) = f(0) \Rightarrow e^a = e^{2/3} = b$$

$\Rightarrow \quad a = \dfrac{2}{3}$ and $b = e^{2/3}$.

The correct option is (A)

III. For f to be continuous at $x = \frac{\pi}{4}$, we must have

$$f\frac{\pi}{4} = \lim_{n \to \frac{\pi}{4}} (\sin 2x)\tan^2 2x$$

$$= \lim_{n \to \frac{\pi}{4}} \frac{\sin 2x}{\cot^2}$$

$$= \lim_{n \to \frac{\pi}{4}} \frac{2\cos 2x}{-2\cot 2x \operatorname{cosec}^2 2x \cdot 2}$$

$$= \lim_{n \to \frac{\pi}{4}} -\frac{1}{2} \cdot \sin m^3 2x = -\frac{1}{2}.$$

The correct option is (D)

IV. For f to be continuous at $x = 0$ we must have

$$f(0) = \lim_{x\to 0} \frac{\sqrt[3]{1+x} - \sqrt[4]{1+x}}{x}$$

$$= \lim_{x\to 0} \frac{\frac{1}{3(1+x)^{2/3}} - \frac{1}{4(1+x)^{3/4}}}{1}$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

The correct option is (B)

107. **I.** Since $f(x)$ is continuous in [0, 1], therefore,

$$\lim_{n\to\infty} f\left(\frac{\sqrt{n}}{2\sqrt{n}+1}\right) = f\left(\lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n}+1}\right)$$

$$= f\left(\frac{1}{2}\right) = 2.$$

The correct option is (B)

II. Since f is continuous on [2, 5], therefore f assumes at least once, every values between $f(2)$ and $f(5)$. But it is given that $f(x)$ takes only rational values for all x and there are irrational values also between $f(2)$ and $f(5)$, this is possible only if $f(x)$ has a constant rational value at all points between $x = 2$ and $x = 5$. Since $f(4) = 8$, $\therefore f(3{\cdot}7) = 8$.

The correct option is (D)

III. Let $g(x) = x^3 - 3$, then $g(x)$ is an increasing function on the interval (1, 2). Since $g(1) = -2$ and $g(2) = 5$, therefore between -2 and 5 there are **6** points where $f(x)$ is discontinuous (as $[x^3 - 3]$ is discontinuous at the points where $x^3 - 3$ is an integer).

The correct option is (A)

IV. $3 \le 3 + 2\cos x \le 5$ for $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$f(x) = [3 + 2\cos x]$ is discontinuous at those points where $3 + 2\cos x$ is an integer.

Now, $3 + 2\cos x = 3$ if $\cos x = 0$. So, $x = \frac{-\pi}{2}, \frac{\pi}{2}$ (not possible)

$3 + 2\cos x = 4$ if $\cos x = \frac{1}{2}$

So, x has two values $\frac{\pi}{3}$ and $\frac{-\pi}{3}$

$3 + 2\cos x = 5$ if $\cos x = 1$. So, $x = 0$

$\therefore$ The number of values of $x = 2 + 1 = 3$

The correct option is (C)

108. **I.** $f(x) = |\,2\,\text{sgn}\,2x\,| + 2$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} f(0+h)$$

$$= |\,2\times 1\,| + 2 = 4$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} |2\,\text{sgn}\,(-2h)| + 2$$

$$= |\,2(-1)\,| + 2 = 4$$

$$\therefore \lim_{x\to 0} f(x) = 4 \ne f(0)$$

$[\because f(0) = |\,2\,\text{sgn}\,(2.0)\,| + 2 = 0 + 2 = 2]$

$\therefore f(x)$ has a removable discontinuity at $x = 0$

The correct option is (C)

II. $$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} \tan\frac{\pi x}{2} = \tan\frac{\pi}{2} = \infty$$

$\therefore \lim_{x\to 1^-} f(x)$ does not exist

$\therefore f(x)$ has infinite discontinuity at $x = 1$.

The correct option is (B)

III. For any $x \ne 0$, $-1 \le \sin\frac{1}{x} \le 1$, but as $x \to 0$, $\sin\frac{1}{x}$ does not approach to any particular value but oscillates between -1 and 1.

The correct option is (A)

IV. $$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{|x+2|}{\tan^{-1}(x+2)}$$

$$= \lim_{x\to -2} \frac{x+2}{\tan^{-1}(x+2)} = 1$$

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{|x+2|}{\tan^{-1}(x+2)}$$

$$= \lim_{x\to -2} \frac{-(x+2)}{\tan^{-1}(x+2)} = 1$$

$\therefore$ Both the limits exist but are unequal

$\therefore f(x)$ has jump discontinuity at $x = -2$.

The correct option is (D)

109. **I.** Let $f(x) = \sin x - x + 1$.

$f(0) = 1 > 0$ and $f\left(\frac{3\pi}{2}\right) = -1 - \frac{3\pi}{2} + 1 = -\frac{3\pi}{2} < 0$.

Thus, by intermediate value theorem, there is a $x \in \left(0, \frac{3\pi}{2}\right)$ such that $f(x) = 0$.

Similarly, argue for II, III and IV.

The correct option is (C)

Assertion-Reason Type

110. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \dfrac{f(x+h)-f(x+0)}{h}$

$= \lim_{h\to 0} \dfrac{f(x)\cdot f(h)-f(x)\cdot f(0)}{h}$

$= \lim_{h\to 0} \dfrac{f(h)-f(0)}{h}\cdot f(x)$

$= f'(0)\cdot f(x) = 2f(x).$ $\quad (\because f'(0)=2)$

Now, $\dfrac{df}{dx} = 2f$ or $\dfrac{df}{f} = 2\,dx \Rightarrow d(\log f - 2x) = 0$

$\therefore \log f - 2x = c$, \f $= e^{2x+c} = e^c\cdot e^{2x} = Ae^{2x}$,

where $A = e^c$ = constant.

The correct option is (C)

111. **(a).** If $x < -1$, then $x > x^3$. So, $f(x) = x$

If $x = -1$, then $x = x^3$. So, $f(x) = x$

If $-1 < x < 0$, then $x < x^3$. So, $f(x) = x^3$

If $x = 0$, then $x = x^3$. So, $f(x) = x^3$

If $0 < x < 1$, then $x > x^3$. So, $f(x) = x$

If $x = 1$, then $x = x^3$. So, $f(x) = x$

If $x > 1$, then $x < x^3$. So, $f(x) = x^3$

Thus, $f(x) = x, x \le -1$

$f(x) = x^3, -1 < x \le 0$

$f(x) = x, 0 < x \le 1$

$f(x) = x^3, x > 1$

Clearly, ***f*(*x*) is not differentiable at *x* = – 1, 0, 1.**

The correct option is (A)

112. Let $h(x) = |x|$ for all x. Clearly, $h(x)$ is continuous for all x.

Then, $g(x) = |f(x)| = h[f(x)] = (hof)(x)$ for all x.

Since composition of two continuous functions is continuous, therefore, ***g* is continuous if *f* is continuous.**

The correct option is (A)

113. $f'(x) = \dfrac{-1}{\sqrt{1-\left(\dfrac{2x}{1+x^2}\right)^2}} \times \dfrac{d}{dx}\left(\dfrac{2x}{1+x^2}\right)$

$= \dfrac{-(1+x^2)}{\sqrt{(1+x^2)^2-4x^2}} \times \dfrac{2(1-x^2)}{(1+x^2)^2}$

$= \dfrac{-2}{1+x^2}\cdot\dfrac{1-x^2}{|1-x^2|} = \begin{cases} \dfrac{-2}{1+x^2}, & \text{if } |x|<1 \\ \dfrac{2}{1+x^2}, & \text{if } |x|>1 \end{cases}$

Clearly, $f(x)$ is differentiable everywhere except at the points where $|x| = 1$ i.e., $x = \pm 1$.

Hence, ***f*(*x*) is differentiable on (– ∞, ∞) \ { – 1, 1}.**

The correct option is (D)

114. We have,

$f(x) = \lim_{n\to\infty} \sum_{r=1}^{n} \dfrac{[2rx]}{n^2} = \lim_{n\to\infty} \sum_{r=1}^{n} \dfrac{2rx}{n^2} - \lim_{n\to\infty} \sum_{r=1}^{n} \dfrac{[2rx]}{n^2}$

Now, we have $0 \le \{2rx\} < 1$

i.e., $\sum_{r=1}^{n} 0 \le \sum_{r=1}^{n} \{2rx\} < \sum_{r=1}^{n} 1$

i.e., $\dfrac{0}{n^2} \le \sum_{r=1}^{n} \dfrac{\{2rx\}}{n^2} < \dfrac{n}{n^2}$

i.e., $0 \le \lim_{n\to\infty} \sum_{r=1}^{n} \dfrac{\{2rx\}}{n^2} < \lim_{n\to\infty} \dfrac{1}{n} = 0$

$\Rightarrow \lim_{n\to\infty} \sum_{r=1}^{n} \dfrac{\{2rx\}}{n^2} = 0$

{By Sandwich Theorem}

Thus, we have $f(x) = 2x\cdot \lim_{n\to\infty} \dfrac{1}{n^2}\sum_{r=1}^{n} r$

$= 2x\cdot \lim_{n\to\infty} \dfrac{n(n+1)}{2n^2} = 2x\cdot\dfrac{1}{2} = x$

Thus, $f(x)$ is continuous everywhere.

The correct option is (A)

115. We have,

$f(x) = \lim_{n\to\infty} \dfrac{\cos(\pi x) - x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}}$

i.e., $f(x) = \dfrac{\cos(\pi x)-0}{1+x-0}, |x/<1$

$= \dfrac{\cos(\pi x)-\sin(x-1)}{1+1-1}, \ |x| = 1$

$= \lim_{n\to\infty} \dfrac{\dfrac{\cos(\pi x)}{x^{2n}} - \sin(x-1)}{\dfrac{1}{x^{2n}} + x - 1}$

$= \dfrac{-\sin(x-1)}{x-1}, \ |x| > 1$

i.e., $f(x) = \dfrac{\cos(\pi x)}{1+x}, \ |x| < 1$

$= -1 + \sin 2, \ x = -1$

$= -1, \quad x = 1$

$$= -\frac{\sin(x-1)}{x-1}, \ |x| > 1$$

At $x = -1$, we have,

$$\lim_{h\to 0} f(-1+h) = \lim_{h\to 0} \frac{\cos\pi(-1+h)}{1+(-1+h)}$$

$$= \lim_{h\to 0} \frac{-\cos(\pi h)}{h} = -\infty$$

$f(-1) = -1 + \sin 2$

implies discontinuity at $x = -1$

At $x = 1$, we have,

$$\lim_{h\to 0} f(1+h) = \lim_{h\to 0} \frac{-\sin(1+h-1)}{1+h-1}$$

$$= \lim_{h\to 0} \frac{-\sin h}{1+(1-h)} = \frac{-1}{2}$$

$$\lim_{h\to 0} f(1-h) = \lim_{h\to 0} \frac{\cos\pi(1-h)}{1+(1-h)} = \frac{-1}{2}$$

$f(1) = -1$

implies discontinuity at $x = 1$

The correct option is (A)

116. We have,

$f(x) = sgn(x) = -1, x < 0$
$= 0, x = 0$
$= 1, x > 0$

and, $g(x) = x(1 - x^2)$

Now, $fog(x) = -1, x(1 - x^2) < 0$
$= 0, x(1 - x^2) = 0$
$= 1, x(1 - x^2) > 0$

Solving the inequality

$x(1 - x^2) < 0$

i.e., $x(x - 1)(x + 1) > 0$ i.e., $x \in (-1, 0) \cup (1, \infty)$

Thus, we have,

$$fog(x) = \begin{cases} -1, & x \in (-1, 0) \cup (1, \infty) \\ 0, & x \in \{-1, 0, 1\} \\ 1, & x \in (-\infty, -1) \cup (1, 0) \end{cases}$$

which is continuous everywhere except at $x \in \{-1, 0, 1\}$

Also,

$gof(x) = f(1 - f^2) = -1\,[1 - (-1)^2], x < 0$
$= 0\,(1 - 0^2), x = 0$
$= 1\,(1 - 1^2), x > 0$

i.e., $gof(x) = 0, \forall x \in R$

which is continuous everywhere.

The correct option is (D)

117. We know,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(x\left(1+\frac{h}{x}\right)\right) - f(x)}{h}$$

$$\Rightarrow \quad f'(1) = \lim_{h\to 0} \frac{f(x)\cdot f\left(1+\frac{h}{x}\right) - f(x)}{h}$$

[given $f(xy) = f(x)\cdot f(y)$]

$$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{f(x)\cdot\left\{1+\frac{h}{x}\left(1+g\left(h/x\right)\right)\right\} - f(x)}{h}$$

[given $f(1 + x) = 1 + x(1 + g(x))$]

$$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{f(x)\left\{1+\frac{h}{x}\left(1+g\left(h/x\right)\right) - 1\right\}}{h}$$

$$\Rightarrow \quad f'(x) = \frac{f(x)}{x} \qquad (1)$$

$\left[\text{as } \lim_{h\to 0} g(x) = 0\right]$

$$\therefore \int_1^2 \frac{f(x)}{f'(x)}\cdot\frac{1}{1+x^2}\,dx = \int_1^2 \frac{x}{1+x^2}\,dx \qquad [\text{using (1)}]$$

$$= \frac{1}{2}\left(\log|1+x^2|\right)_1^2$$

$$= \frac{1}{2}\left[\log\left(5/2\right)\right]$$

The correct option is (A)

118. We have, $y = t^2 + t\,|t|$ and $x = 2t - |t|$

When $t \geq 0$

$x = 2t - t = t, y = t^2 + t^2 = 2t^2$

$\therefore \quad x = t$ and $y = 2t^2$

$\Rightarrow \quad y = 2x^2 \ \forall x \geq 0$

Also, when $t < 0$

$x = 2t + t = 3t$ and $y = t^2 - t^2 = 0$

$\Rightarrow \quad y = 0$ for all $x < 0$

Hence, $f(x) = \begin{cases} 2x^2, x \geq 0 \\ 0\ , x < 0 \end{cases}$ which is clearly continuous for all x.

The correct option is (A)

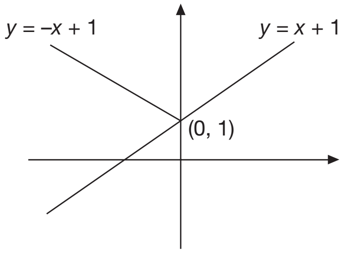
Previous Year's Questions

119. $f(x)=\dfrac{1-\tan x}{4x-\pi} \Rightarrow f(\frac{\pi}{4}) = \lim\limits_{x\to\frac{\pi}{4}} \dfrac{1-\tan x}{4x-\pi} = -\dfrac{1}{2}$

The correct option is (C)

120. Given that $f(x) = \min\{x + 1, |x| + 1\}$

Now, $f(x) = x + 1\ \forall\ x \in$ R.



The correct option is (C)

121. $f(0) = \lim\limits_{x\to 0} \dfrac{1}{x} - \dfrac{2}{e^{2x}-1}$

$= \lim\limits_{x\to 0} \dfrac{e^{2x}-1-2x}{x(e^{2x}-1)}$

$= \lim\limits_{x\to 0} \dfrac{2e^{2x}-2}{(e^{2x}-1)+2xe^{2x}}$

$= \lim\limits_{x\to 0} \dfrac{4e^{2x}}{4e^{2x}+4xe^{2x}} = 1$

The correct option is (D)

122. Using differentiation formula, we write

$f'(1) = \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$

$\Rightarrow f'(1) = \lim\limits_{h\to 0} \dfrac{(1+h-1)\sin\left(\dfrac{1}{1+h-1}\right)-0}{h} = \lim\limits_{h\to 0} \dfrac{h}{h}\sin\left(\dfrac{1}{h}\right)$

$\Rightarrow f'(1) = \lim\limits_{h\to 0} \sin\left(\dfrac{1}{h}\right)$

$\therefore f$ is not differentiable at $x = 1$.

Similarly, $f'(0) = \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h}$

$\Rightarrow f'(0) = \lim\limits_{h\to 0} \dfrac{(h-1)\sin\left(\dfrac{1}{h-1}\right)-\sin(1)}{h}$

$\Rightarrow f$ is also not differentiable at $x = 0$.

The correct option is (A)

123. xRy need not implies yRx

$S : \dfrac{m}{n} s \dfrac{p}{q} \Leftrightarrow qm = pn$

$\dfrac{m}{n} s \dfrac{m}{n}$ reflexive

$\dfrac{m}{n} s \dfrac{p}{q} \Rightarrow \dfrac{p}{q} s \dfrac{m}{n}$ symmetric

$\dfrac{m}{n} s \dfrac{p}{q}, \dfrac{p}{q} s \dfrac{r}{s}$

$\Rightarrow qm = pn, ps = rq$

$\Rightarrow ms = rn$ transitive

S is an equivalence relation.

The correct option is (B)

124. $\lim\limits_{x\to 0} \dfrac{\sin(p+1)+\sin x}{x} = q = \lim\limits_{x\to 0} \dfrac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}}$

$\lim\limits_{x\to 0} (p+1)\cos(p+1)\times + \cos x = q = \dfrac{1}{2}$

$\Rightarrow p+1+1 = \dfrac{1}{2} \Rightarrow p - \dfrac{3}{2}; q = \dfrac{1}{2}$

The correct option is(B)

125. $f(x) = [x]\cos\left(\dfrac{2x-1}{2}\right)\pi = [x]\cos\left(x-\dfrac{1}{2}\right)\pi$

$= [x] \sin\ \pi x$ is continuous for every real x.

The correct option is (A)

126. $f(x) = 7 - 2x; x < 2$

$= 3; 2 \le \times \le 5$

$= 2x - 7; x > 5$

$f(x)$ is constant function in [2, 5]

f is continuous in [2, 5] and differentiable in (2, 5) and $f(2) = f(5)$

by Rolle's theorem $f'(4) = 0$

$\therefore$ Statement 2 and statement 1 both are true and statement 2 is correct explanation for statement 1.

The correct option is(B)

127. For $f(x)$ to be continuous

$2k = 3m + 2$

$\Rightarrow 2k - 3m = 2$ (1)

Also, for $f(x)$ to be differentiable

$\dfrac{k}{4} = m$

$\Rightarrow k = 4m.$

from (1), $8m - 3m = 2$

$\Rightarrow 5m = 2$

$\Rightarrow m = \dfrac{2}{5}$

$\therefore k = 4 \times \dfrac{2}{5} = \dfrac{8}{5}$ $\qquad \therefore k + m = \dfrac{2}{5} + \dfrac{8}{5} = \dfrac{10}{5} = 2$.

The correct option is (D)

128. In the neighbourhood of $x = 0, f(x) = \log 2 - \sin x$

$\therefore \quad g(x) = f(f(x)) = \log 2 - \sin(f(x))$

$\log 2 - \sin(\log 2 - \sin x)$

Since $g(x)$ is differentiable at $x = 0$,

$\therefore \quad g'(x) = -\cos(\log 2 - \sin x)\,(-\cos x)$

$\Rightarrow \quad g'(0) = \cos(\log 2)$

The correct option is (C)