CHAPTER 14

Applications of Derivatives

Chapter Highlights

Tangents and normals, Slope of tangent, Equation of tangent, Slope of normal, Equation of normal, Angle of intersection of two curves, Length of tangent, length of normal, sub-tangent and subnormal, Length of intercept made on axes by the tangent, Length of perpendicular from origin to the tangent, Increasing and decreasing functions (monotonicity), Test for monotonicity of functions, Maxima and minima of functions, Method to determine the points of local maxima and local minima, Greatest and least values of a function in a closed interval (absolute maximum and absolute minimum), Concavity and convexity of a function, Point of inflexion, Rolle's and lagrange's mean value theorem, Rolle's theorem, Lagrange's mean value theorem, Application of $\frac{dy}{dx}$ as a rate measure.

TANGENTS AND NORMALS

Geometrical Meaning of Derivative at a Point

The derivative of a function $f(x)$ at a point $x = a$ is the slope of the tangent to the curve $y = f(x)$ at the point $[a, f(a)]$.

SLOPE OF TANGENT

Consider a curve $y = f(x)$ and a point $P(x, y)$ on this curve. If tangent to the curve at $P(x, y)$ makes an angle θ with the positive direction of x-axis, then, at the point $P(x, y)$: $\frac{dy}{dx}$ $= \tan\theta = m =$ gradient or slope of tangent to the curve at $P(x, y)$.

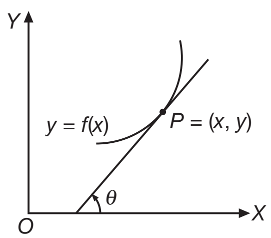

Fig. 14.1

EQUATION OF TANGENT

The equation of a tangent to a curve $y = f(x)$ at a given point $P\ (x_1, y_1)$ is given by

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} (x - x_1)$$

[Using point slope form of equation of the straight line]

SLOPE OF NORMAL

The normal to a curve at a point $P(x_1, y_1)$ is a line perpendicular to the tangent at P and passing through P. Slope of the normal

$$= \frac{-1}{\text{Slope of tangent}} = \frac{-1}{\left(\frac{dy}{dx}\right)_{P(x_1 \cdot y_1)}} = -\left(\frac{dx}{dy}\right)_{P(x_1,\, y_1)}$$

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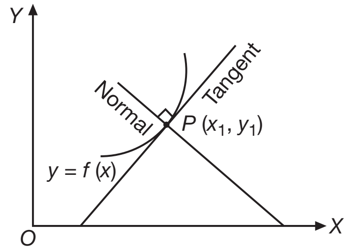


Fig. 14.2

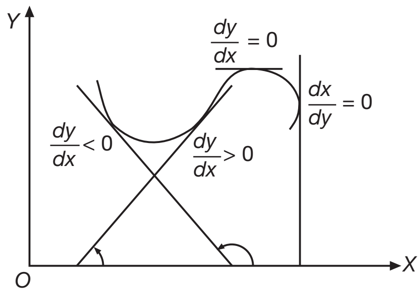


Fig. 14.3

TRICK(S) FOR PROBLEM SOLVING

- If $\frac{dy}{dx} > 0$, the tangent makes an acute angle with the x-axis.
- If $\frac{dy}{dx} < 0$, the tangent makes an obtuse angle with the x-axis.
- If $\frac{dy}{dx} = 0$, the tangent is parallel to x-axis.
- If the tangent is perpendicular to x-axis, then $\frac{dy}{dx} = \infty$, i.e., $\frac{dy}{dx} = 0$.
- If the tangent is equally inclined to the axes, then $\frac{dy}{dx} = \tan 45° \text{ or } \tan 135° = \pm 1$.
- The slope of a line having equation $ax + by + c = 0$ is given by $m = -\frac{a}{b} = -\frac{\text{coefficient of } x}{\text{coefficient of } y}$
- The two lines having slopes m_1 and m_2 are
  (i) perpendicular if $m_1 m_2 = -1$ and
  (ii) parallel if $m_1 = m_2$.

Thus, if m be the slope of a line, then the slope of a line perpendicular to it is $-\frac{1}{m}$ and parallel to it is m.

SOLVED EXAMPLES

1. If the tangent at each point of the curve

$$y = \frac{2}{3}x^3 - 2ax^2 + 2x + 5$$

makes an acute angle with the positive direction of x-axis, then

(A) $a \geq 1$ (B) $-1 \leq a \leq 1$
(C) $a \leq -1$ (D) None of these

Solution: (B)

We have, $y = \frac{2}{3}x^3 - 2ax^2 + 2x + 5$

$$\Rightarrow \quad \frac{dy}{dx} = 2x^2 - 4ax + 2$$

Since, the tangent makes an acute angle with the positive direction of x-axis, therefore,

$$\frac{dy}{dx} \geq 0 \Rightarrow 2x^2 - 4ax + 2 \geq 0 \text{ for all } x$$

$$\Rightarrow 16a^2 - 16 \leq 0$$

$$(\because \text{ Disc.} = (4a)^2 - 4\,(2)\,(2) \leq 0)$$

$$\Rightarrow a^2 - 1 \leq 0 \text{ i.e. } (a-1)(a+1) \leq 0$$

$$\Rightarrow -1 \leq a \leq 1$$

2. If m be the slope of a tangent to the curve $e^{2y} = 1 + 4x^2$, then

(A) $m < 1$ (B) $|m| \leq 1$
(C) $|m| > 1$ (D) None of these

Solution: (B)

We have, $e^{2y} = 1 + 4x^2 \quad \Rightarrow e^{2y} \cdot 2\frac{dy}{dx} = 8x$

$$\Rightarrow \frac{dy}{dx} = \frac{4x}{e^{2y}}$$

$$= \frac{4x}{1+4x^2}$$

$$\therefore \text{ Slope of tangent} = m = \frac{4x}{1+4x^2}$$

$$\Rightarrow |m| = \frac{4|x|}{1+4|x|^2} \leq 1$$

$$\left[\because (1-2|x|)^2 \geq 0 \quad \Rightarrow 1+4|x|^2 - 4|x| \geq 0 \quad \Rightarrow \frac{4|x|}{1+4|x|^2} \leq 1\right]$$

3. The tangent to the curve $\sqrt{x}+\sqrt{y} = 4$ is equally inclined to the axes at the point

(A) (1, – 2) (B) (4, 4)
(C) (4, – 4) (D) (– 4, 4)

Solution: (B)

We have, $\sqrt{x}+\sqrt{y} = 4$ (1)

$$\Rightarrow \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0, \text{ or } \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}.$$

Since the tangent is equally inclined to the axes,

$\frac{dy}{dx}$ = tan 45° or tan 135° i.e., 1 or – 1. Thus

$-\frac{\sqrt{y}}{\sqrt{x}} = \pm 1$. This gives $y = x$

From (1), $\sqrt{x}+\sqrt{x} = 4 \Rightarrow x = 4$. Also, $y = x = 4$.

The point is (4, 4).

4. The angle between the tangents to the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points $(a, 0)$ and $(0, b)$ is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$

(C) $\frac{\pi}{3}$ (D) None of these

Solution: (B)

We have, $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

$$\Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-b^2x}{a^2y}$$

$$\therefore \tan\theta_1 = \left.\frac{dy}{dx}\right]_{(a,\,0)} = \infty \Rightarrow \theta_1 = \frac{\pi}{2}$$

and $$\tan\theta_2 = \left.\frac{dy}{dx}\right]_{(0,\,b)} = 0 \Rightarrow \theta_2 = 0$$

Hence, the angle between the two tangents is

$$\theta = \theta_1 - \theta_2 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

5. The equation of the normal to the curve $y = e^{-2|x|}$ at the point where the curve cuts the line $x = \frac{1}{2}$ is

(A) $2e(ex + 2y) = e^2 - 4$ (B) $2e(ex - 2y) = e^2 - 4$
(C) $2e(ey - 2x) = e^2 - 4$ (D) None of these

Solution: (B)

At the point $x = \frac{1}{2}$, $y = e^{-1}$

Since, $x = \frac{1}{2} > 0$, $\therefore y = e^{-2x}$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -2e^{-2x} \Rightarrow \left.\frac{dy}{dx}\right]_{\left(\frac{1}{2},\frac{1}{e}\right)} = -2e^{-1} = \frac{-2}{e}$$

Thus, the equation of normal is

$$\left(y-\frac{1}{e}\right) = \frac{e}{2}\left(x-\frac{1}{2}\right) \text{ i.e., } 2e(ex - 2y) = e^2 - 4$$

6. If the line $ax + by + c = 0$ is a tangent to the curve $xy = 4$, then

(A) $a > 0, b > 0$ (B) $a > 0, b < 0$
(C) $a < 0, b > 0$ (D) $a < 0, b < 0$

Solution: (A, D)

We have, $xy = 4 \Rightarrow x\cdot\frac{dy}{dx} + y\cdot 1 = 0$

i.e., $$\frac{dy}{dx} = -\frac{y}{x} = -\frac{4}{x^2} \qquad (\because xy = 4)$$

$\therefore$ Slope of tangent $= -\frac{4}{x^2}$

Slope of the line $ax + by + c = 0$ is $= -\frac{a}{b}$.

Since the given line is a tangent to the curve

$$\therefore \quad -\frac{4}{x^2} = -\frac{a}{b} \Rightarrow \frac{a}{b} > 0$$

It is possible only when $a > 0, b > 0$ or $a < 0, b < 0$.

7. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then

(A) $a > 0, b > 0$ (B) $a > 0, b < 0$
(C) $a < 0, b > 0$ (D) $a < 0, b < 0$

Solution: (B, C)

We have $xy = 1 \Rightarrow y = \frac{1}{x}$

$$\therefore \frac{dy}{dx} = -\frac{1}{x^2}$$

$\therefore$ The slope of the normal $= x^2$

If $ax + by + c = 0$ is normal to the curve $xy = 1$

then $x^2 = -\frac{a}{b}$ $\quad \therefore -\frac{a}{b} > 0$

$\Rightarrow a > 0, b < 0$ or $a < 0, b > 0$

8. The line $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y = be^{-x/a}$ at the point

(A) $(-a, ba)$ (B) $\left(a, \frac{a}{b}\right)$

(C) $\left(a, \frac{b}{a}\right)$ (D) None of these

Solution: (D)

We have, $y = be^{-x/a}$ (1)

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{b}{a}e^{-x/a}$$

Since the line $\frac{x}{a}+\frac{y}{b}=1$ touches (1)

$$\therefore \quad \frac{-1/a}{1/b} = -\frac{b}{a}e^{-x/a} \Rightarrow -\frac{b}{a} = -\frac{b}{a}e^{-x/a}$$

$$\Rightarrow \quad 1 = e^{-x/a} \Rightarrow -\frac{x}{a} = 0 \text{ i.e., } x = 0.$$

$$\therefore \quad y = be^0 = b.$$

Hence, the required point is $(0, b)$.

9. The curve $y = ax^3 + bx^2 + cx$ is inclined at 45° to x-axis at (0, 0) but it touches x-axis at (1, 0), then the values of a, b, c are given by

(A) $a = 1, b = -2, c = 1$
(B) $a = 1, b = 1, c = -2$
(C) $a = -2, b = 1, c = 1$
(D) $a = -1, b = 2, c = 1$

Solution: (A)

We have, $y = ax^3 + bx^2 + cx$

$$\Rightarrow \quad \frac{dy}{dx} = 3ax^2 + 2bx + c$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{(0,0)} = c = \tan 45° = 1 \text{ (Given)}$$

$$\Rightarrow \quad c = 1$$

Also, $\left.\frac{dy}{dx}\right]_{(1,0)} = 3a + 2b + c = 0$

($\because$ x-axis is tangent at (1, 0))

$$\Rightarrow \quad 3a + 2b + 1 = 0$$

which is true if $a = 1, b = -2$.

Hence, $a = 1, b = -2, c = 1$.

10. If the line $y = 2x$ touches the curve $y = ax^2 + bx + c$ at the point where $x = 1$ and the curve passes through the point $(-1, 0)$, then the values of a, b and c are

(A) $a = \frac{1}{2}, b = 1, c = \frac{1}{2}$

(B) $a = 1, b = \frac{1}{2}, c = \frac{1}{2}$

(C) $a = \frac{1}{2}, c = \frac{1}{2}, b = 1$

(D) None of these

Solution: (A)

The given curve is $y = ax^2 + bx + c$ (1)

Since the point $(-1, 0)$ lie on it

$$\therefore \quad a - b + c = 0 \quad (2)$$

Also, $y = 2x$ is a tangent to (1) at $x = 1$, so that $y = 2$.

Since the point (1, 2) lies on (1),

$$\therefore \quad a + b + c = 2 \quad (3)$$

Also $\left.\frac{dy}{dx}\right]_{(1,2)} = (2ax + b)]_{(1,2)} = 2,$

$$\therefore \quad 2a + b = 2 \quad (4)$$

Solving (2), (3) and (4): $a = \frac{1}{2}, b = 1, c = \frac{1}{2}$

ANGLE OF INTERSECTION OF TWO CURVES

Let $y = f(x)$ and $y = g(x)$ be two curves intersecting at a point $P(x_1, y_1)$. Then, the angle of intersection of these two curves is defined as the angle between the tangents to the two curves at their point of intersection.

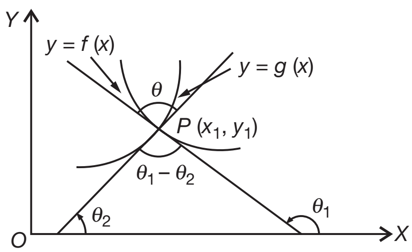


Fig. 14.4

If θ is the required angle of intersection, then,

$$\theta = \pm(\theta_1 - \theta_2),$$

where θ_1 and θ_2 are the inclinations of tangents to the curves $y = f(x)$ and $y = g(x)$ respectively at the point P.

TRICK(S) FOR PROBLEM SOLVING

Short-Cut Method to Find the Angle of Intersection

- Find $f'(x)$ and $g'(x)$.
- If $f'(x) \times g'(x) = -1$, then the two curves are said to cut each other orthogonally, wherever they cut.
- If the product is not -1, solve the equation of the two curves to get their point of intersection. If (θ, θ) be their point of intersection, then find $f'(\theta)$ and $g'(\theta)$. Let $m_1 = f'(\theta)$ and $m_2 = g'(\theta)$.
- If θ is the angle between the tangents, then

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2} = \pm\frac{f'(\alpha) - g'(\alpha)}{1 + f'(\alpha)g'(\alpha)}$$

Repeat this process for other points of intersection.

IMPORTANT POINTS

The two curves are said to touch each other at their point of intersection (α, β), if the slope of their tangents at (α, β) are equal.

SOLVED EXAMPLES

11. The two curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ at the point (1, 2)

(A) Intersect orthogonally

(B) Intersect at an angle $\frac{\pi}{3}$

(C) Touch each other

(D) None of these

Solution: (C)

We have, $y^2 = 4x$ (1)

and $x^2 + y^2 - 6x + 1 = 0$ (2)

Differentiating (1) with respect to x, we get

$$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{(1,\,2)} = \frac{2}{2} = 1 = m_1 \text{ (say)}$$

Differentiating (2) with respect to x, we get

$$2x + 2y\frac{dy}{dx} - 6 = 0 \Rightarrow \frac{dy}{dx} = \frac{3-x}{y}$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{(1,\,2)} = \frac{3-1}{2} = 1 = m_2 \text{ (say)}$$

Since $m_1 = m_2$, therefore the two curves touch each other at (1, 2).

12. The angle of intersection of the curves $y = 2\sin^2 x$ and $y = \cos 2x$ at $x = \frac{\pi}{6}$ is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$ (D) None of these

Solution: (B)

We have,

$$y = 2\sin^2 x \quad (1)$$

and

$$y = \cos 2x \quad (2)$$

Differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = 4\sin x\cos x$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{x=\frac{\pi}{6}} = 4\cdot\left(\frac{1}{2}\right)\frac{\sqrt{3}}{2}\cdot = \sqrt{3} = m_1 \text{ (say)}$$

Differentiating (2) with respect to x, we get

$$\frac{dy}{dx} = -2\sin 2x \Rightarrow \left.\frac{dy}{dx}\right]_{x=\frac{\pi}{6}} = -2\sin\frac{\pi}{3}$$

$$= -\sqrt{3} = m_2 \text{ (say).}$$

Hence, angle between the two curves is

$$\theta = \pm\tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) = \pm\tan^{-1}\sqrt{3} = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

13. The line $\frac{x}{a} + \frac{y}{b} = 2$ touches the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ at the point (a, b) for

(A) $n = 2$ only (B) $n = -3$ only

(C) n is any real number (D) None of these

Solution: (C)

We have, $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

$$\Rightarrow \quad \frac{nx^{n-1}}{a^n} + \frac{ny^{n-1}}{b^n}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{(a,\,b)} = -\frac{b^n \cdot a^{n-1}}{a^n b^{n-1}} = -\frac{b}{a}$$

$\therefore$ The equation of tangent at (a, b) is

$$y - b = -\frac{b}{a}(x - a) \Rightarrow \frac{x}{a} + \frac{y}{b} = 2.$$

$\therefore$ The line $\frac{x}{a} + \frac{y}{b} = 2$ touches the curve at (a, b), for all n.

14. The least value of a for which $\frac{4}{\sin x} + \frac{1}{1 - \sin x} = a$ has at least one solution in the interval $\left(0, \frac{\pi}{2}\right)$ is

(A) 9 (B) 8
(C) 6 (D) 4

Solution: (A)

$$f(x) = \frac{4}{\sin x} + \frac{1}{1 - \sin x} = a$$

$$\Rightarrow \quad f'(x) = -\frac{4\cos x}{\sin^2 x} + \frac{\cos x}{(1 - \sin x)^2}$$

$$= \cos x\left(\frac{1}{(1 - \sin x)^2} - \frac{4}{\sin^2 x}\right)$$

$$\therefore \quad f'(x) = 0$$

$$\Rightarrow \quad \frac{1}{(1 - \sin x)^2} - \frac{4}{\sin^2 x} = 0 \text{ as } \cos x \neq 0 \text{ in } \left(0, \frac{\pi}{2}\right)$$

The given $x = 2/3$. Substituting in $f(x) = 0$, we get $a = 9$.

15. The curves $y^2 = 2x$ and $2xy = k$ cut at right angles if

(A) $k^2 = 8$ (B) $k^2 = 4$
(C) $k^2 = 2$ (D) None of these

Solution: (A)

Let $P(x_1, y_1)$ be the point of intersection of the two curves.
We have,

$$y^2 = 2x \Rightarrow 2y\frac{dy}{dx} = 2 \Rightarrow m_1 = \left(\frac{dy}{dx}\right)\Big]_{(x_1, y_1)}$$

$$= \frac{1}{y_1}$$

and $\quad 2xy = k \quad \Rightarrow \quad x\frac{dy}{dx} + y = 0$

$$\Rightarrow m_2 = \left(\frac{dy}{dx}\right)\Big]_{(x_1, y_1)} = -\frac{y_1}{x_1}$$

Since the two curves intersect at right angles,

$$\therefore \quad m_1 m_2 = -1 \Rightarrow \left(\frac{1}{y_1}\right)\left(-\frac{y}{x_1}\right) = -1 \Rightarrow x_1 = 1$$

and hence from $y_1^2 = 2x_1$, we get $y_1^2 = 2$.

Since (x_1, y_1) also lies on $2xy = k$

$$\therefore \quad k^2 = 4x_1^2 y_1^2 = 4 \times 1 \times 2 = 8$$

16. If the curve $y = x^2 + bx + c$ touches the line $y = x$ at the point (1, 1), then the values of x for which the curve has a negative gradient are

(A) $x < \frac{1}{2}$ (B) $x > \frac{1}{2}$
(C) $x < -\frac{1}{2}$ (D) $x > -\frac{1}{2}$

Solution: (A)

We have, $y = x^2 + bx + c$

$$\Rightarrow \quad \frac{dy}{dx} = 2x + b$$

Since the curve touches the line $y = x$ at the point (1, 1)

$$\therefore \quad (2x + b)\big]_{(1,1)} = 1 \text{ i.e., } 2 + b = 1 \Rightarrow b = -1.$$

Also, the curve passes through the point (1, 1)

$$\therefore \quad 1 = 1 + b + c \text{ i.e., } c = -b = 1$$

$$\therefore \quad y = x^2 - x + 1 \Rightarrow \frac{dy}{dx} = 2x - 1$$

Now, $\frac{dy}{dx} < 0 \Rightarrow 2x - 1 < 0 \Rightarrow \boldsymbol{x} < \frac{1}{2}$

REMEMBER

If one angle between the tangents (acute/obtuse) is θ, then the other angle between the tangents (obtuse/acute) is $(180° - \theta)$.
Generally, we take the acute angle to be the angle of intersection of the given curves.

LENGTH OF TANGENT, LENGTH OF NORMAL, SUB-TANGENT AND SUBNORMAL

Let the tangent and normal at the point $P(x, y)$ on the curve meet the axis of x at the points T and N respectively. Let M be the foot of the ordinate at P. Then,

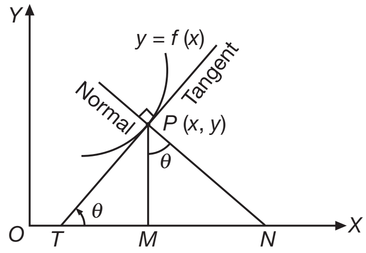


Fig. 14.5

1. Length of the tangent = $PT = |y \operatorname{cosec} \theta|$

$$= \left| y\sqrt{1+\cot^2\theta} \right| = \left| \frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}} \right|$$

2. Length of the normal = $PN = |y \sec \theta|$

$$= \left| y\sqrt{1+\tan^2\theta} \right| = \left| y\sqrt{1+\left(\frac{dy}{dx}\right)^2} \right|$$

3. Subtangent = $TM = |y \cot \theta| = \left| \frac{y}{\left(\frac{dy}{dx}\right)} \right|$

4. Subnormal = $MN = |y \tan \theta| = \left| y\left(\frac{dy}{dx}\right) \right|$.

LENGTH OF INTERCEPT MADE ON AXES BY THE TANGENT

Equation of tangent at any point (x_1, y_1) to the curve $y = f(x)$ is $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} (x - x_1)$.

Then, $$x\text{-intercept} = x_1 - \left(\frac{y_1}{\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}} \right)$$

and $$y\text{-intercept} = y_1 - \left(x_1 \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} \right)$$

SOLVED EXAMPLES

17. The sub-normal at any point of the curve $x^2y^2 = a^2(x^2 - a^2)$ varies as

(A) $(\text{abscissa})^{-3}$ (B) $(\text{abscissa})^3$
(C) $(\text{ordinate})^{-3}$ (D) None of these

Solution: (A)

We have, $x^2y^2 = a^2(x^2 - a^2)$ (1)

$$\Rightarrow \quad x^2 \times 2y\frac{dy}{dx} + y^2 \cdot 2x = a^2 \cdot 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{a^2 - y^2}{xy}$$

$$\therefore \text{ Sub-normal} = y\frac{dy}{dx} = \frac{a^2 - y^2}{x} = \frac{x^2(a^2 - y^2)}{x^3}$$

$$= \frac{a^4}{x^3} \quad [\because \text{ from (1) } x^2(a^2 - y^2) = a^4]$$

⇒ The sub-normal varies inversely as the cube of its abscissa.

18. The sub-tangent at any point of the curve $x^m y^n = a^{m+n}$ varies as

(A) $(\text{abscissa})^2$ (B) $(\text{ordinate})^2$
(C) abscissa (D) ordinate

Solution: (C)

We have, $x^m y^n = a^{m+n}$

$\Rightarrow \quad m \log x + n \log y = (m + n) \log a$

$$\therefore \quad \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = 0 \Rightarrow \frac{dx}{dy} = -\frac{nx}{my}$$

$$\therefore \quad \text{Sub-tangent} = \left| y\frac{dx}{dy} \right| = \frac{nx}{m} \propto x$$

19. For the parabola $y^2 = 4ax$, the ratio of the sub-tangent to the abscissa is

(A) 1 : 1 (B) 2 : 1
(C) $x : y$ (D) $x^2 : y$

Solution: (B)

We have, $y^2 = 4ax$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4a \text{ i.e., } \frac{dy}{dx} = \frac{2a}{y}$$

$$\text{Sub-tangent} = \frac{y}{dy/dx} = \frac{y}{2a/y} = \frac{y^2}{2a} = \frac{4ax}{2a} = 2x.$$

$\therefore$ Sub-tangent : Abscissa = $2x : x = 2 : 1$.

LENGTH OF PERPENDICULAR FROM ORIGIN TO THE TANGENT

Length of perpendicular from origin (0, 0) to the tangent drawn at point $P(x_1, y_1)$ to the curve $y = f(x)$ is

$$p = \left| \frac{y_1 - x_1\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right|$$

INCREASING AND DECREASING FUNCTIONS (MONOTONICITY)

Increasing Function

A function $f(x)$ is said to be an increasing function on an interval I, if

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2),\ \forall\ x_1, x_2 \in I$$

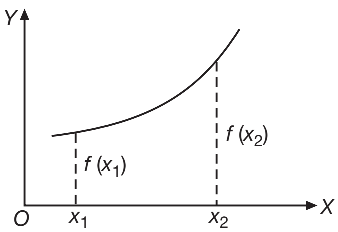


Fig. 14.6 Increasing Function

Strictly Increasing Function

A function $f(x)$ is said to be a strictly increasing function on an interval I, if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2),\ \forall\ x_1, x_2 \in I$$

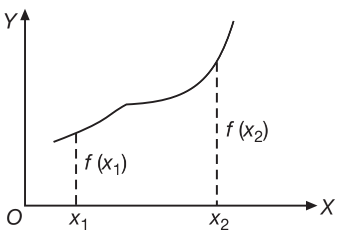


Fig. 14.7 Strictly Increasing Function

Decreasing Function

A function $f(x)$ is said to be a decreasing function on an interval I, if

$$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2),\ \forall\ x_1, x_2 \in I$$

Decreasing Function

Strictly Decreasing Function

A function $f(x)$ is said to be a strictly decreasing function on an interval I, if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)\ \forall\ x_1, x_2 \in I$$

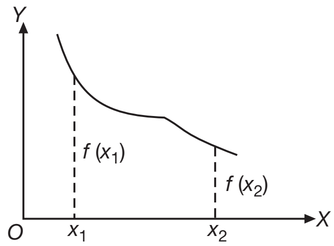


Fig. 14.8 Strictly Decreasing Function

Monotonic Function

A function $f(x)$ is said to be monotonic on an interval I if it is either increasing or decreasing on I.

TEST FOR MONOTONICITY OF FUNCTIONS

1. $f(x)$ is increasing in $[a, b]$ if $f'(x) \ge 0,\ \forall\ x \in [a, b]$.
2. $f(x)$ is strictly increasing in $[a, b]$ if $f'(x) > 0,\ \forall\ x \in [a, b]$.
3. $f(x)$ is decreasing in $[a, b]$ if $f'(x) \le 0,\ \forall\ x \in [a, b]$.
4. $f(x)$ is strictly decreasing in $[a, b]$ if $f'(x) < 0,\ \forall\ x \in [a, b]$.

TRICK(S) FOR PROBLEM SOLVING

- If a function $f(x)$ is strictly increasing (strictly decreasing) on an interval I, then f^{-1} exists and is also strictly increasing (strictly decreasing).
- If $f(x)$ is monotonic on an interval I, then $f(x)$ has at the most one zero in the interval I.
- If the functions $f(x)$ and $g(x)$, both are increasing or decreasing on an interval I, then the composite function $(gof)(x)$ is an increasing function on I.
- If the function $f(x)$ is increasing (decreasing) and $g(x)$ decreasing (increasing) on an interval I, then the composite function $(gof)(x)$ is decreasing on the interval I.
- A function may be increasing in some interval I_1 and decreasing in some other interval I_2.

SOLVED EXAMPLES

20. If $f(x)$ is a polynomial function such that, $f(x) > f(x)$, $\forall\ x \ge 1$ and $f(1) = 0$, then

(A) $f(x) < 0\ \forall\ x \ge 1$ (B) $f(x) \ge 0\ \forall\ x \ge 1$
(C) $f(x) = 0,\ \forall\ x \ge 1$ (D) None of these.

Solution: (B)

We have, $f''(x) > f'(x) > f(x)$, $\forall\, x \geq 1$

$\Rightarrow\; f'(x) - f(x) > 0, \forall\, x \geq 1$

$\Rightarrow\; e^{-x}[f'(x) - f(x)] > 0, \forall\, x \geq 1$

$\Rightarrow\; \dfrac{d}{dx}[e^{-x}f(x)] > 0, \forall\, x \geq 1$

$\therefore\; e^{-x}f(x)$ is an increasing function, $\forall\, x \geq 1$

$\Rightarrow\; e^{-x}f(x) \geq e^{-1}f(1)$ $\quad$ ($\therefore$ $f(x)$ is a polynomial)

$\Rightarrow\; e^{-x}f(x) \geq 0$ $\quad$ [$\therefore$ $f(1) = 0$]

$\Rightarrow\; f(x) \geq 0, \forall\, x \geq 1$

21. The subtangent, ordinate and subnormal to the parabola $y^2 = 4ax$ at a point (different from the origin) are in

(A) GP $\quad$ (b) AP

(C) HP $\quad$ (d) None of these

Solution: (A)

We have, $y^2 = 4ax$

$\Rightarrow\; 2y\dfrac{dy}{dx} = 4a$ i.e., $\dfrac{dy}{dx} = \dfrac{2a}{y}$.

$$\text{Sub-tangent} = \frac{y}{dy/dx} = \frac{y}{2a/y} = \frac{y^2}{2a}$$

$$\text{Sub-normal} = y\frac{dy}{dx} = y \times \frac{2a}{y} = 2a$$

Clearly, $\dfrac{y^2}{2a}$, y, $2a$ are in G.P.

22. The largest term in the sequence

$x_n = \dfrac{n^2}{n^3 + 200}$, $n \in N$, is

(A) $\dfrac{49}{543}$ $\quad$ (B) $\dfrac{8}{89}$

(C) $\dfrac{1}{52}$ $\quad$ (D) None of these

Solution: (A)

Let $\quad f(n) = x_n = \dfrac{n^2}{n^3 + 200}$, $n \in N$

$$\Rightarrow\; f'(n) = \frac{(n^3+200)\cdot 2n - n^2 \cdot 3n^2}{(n^3+200)^2}$$

$$= \frac{n(400 - n^3)}{(n^3+200)^2}$$

But $f'(n) \neq 0$ for any $n \in N$. Hence $f(n)$ has no critical point.

But the function $f(n)$ is increasing for $n < 8$ and it starts decreasing for $n \geq 8$.

Here, $\quad f(7) = \dfrac{49}{543}$ and $f(8) = \dfrac{8}{89}$.

Clearly, $\;f(7) > f(8)$.

Hence, the largest value is $\dfrac{49}{543}$.

23. Let $f(x) = \displaystyle\int_0^x \left|\log_2\{\log_3[\log_4(\cos t + a)]\}\right| dt$ be increasing for all real values of x, then

(A) $a \geq 5$ $\quad$ (B) $0 \leq a < 4$

(C) $a < 0$ $\quad$ (D) None of these

Solution: (A)

We have, $f'(x) = \left|\log_2 \log_3 \log_4(\cos x + a)\right|$

Clearly, $f'(x) \geq 0$ for all x

$\Rightarrow\; x = \pm 1$

$\therefore f(x)$ is increasing for all real x, provided $f(x)$ is defined.

Now, $f(x)$ is defined, if

$\log_3 \log_4(\cos x + a) > 0, \forall\, x \in R$

$\Rightarrow\; \log_4(\cos x + a) > 4° = 1, \forall\, x \in R$

$\Rightarrow\; \cos x + a > 4, \forall\, x \in R$

$\Rightarrow\; a > 4 - \cos x, \forall\, x \in R$

$\Rightarrow\; a > 5$ $\quad$ ($\therefore$ $1 \leq \cos x \leq 1$)

24. Let $f'(\sin x) < 0$ and $f''(\sin x) > 0\; \forall\, x \in \left(0, \dfrac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then $g(x)$ is decreasing in

(A) $\left(0, \dfrac{\pi}{4}\right)$ $\quad$ (B) $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

(C) $\left(0, \dfrac{\pi}{2}\right)$ $\quad$ (D) None of these

Solution: (A)

We have, $g'(x) = f'(\sin x)\cdot\cos x - f'(\cos x)\cdot\sin x$

$$\Rightarrow\; g''(x) = -f'(\sin x)\cdot\sin x + f''(\sin x)\cdot\cos^2 x$$

$$+ f''(\cos x)\cdot\sin^2 x - f'(\cos x)\cdot\cos x$$

$$> 0, \forall\, x \in \left(0, \frac{\pi}{2}\right)$$

$\Rightarrow g'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$ Also, $g'\left(\frac{\pi}{4}\right) = 0$

$\Rightarrow g'(x) > 0\ \forall\ x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $g'(x) < 0\ \forall\ x \in \left(0, \frac{\pi}{4}\right)$

Thus, $g(x)$ is decreasing in $\left(0, \frac{\pi}{4}\right)$

25. Let $f(x) = (1 + b^2)\,x^2 + 2bx + 1$ and m (b) the minimum value of $f(x)$ for a given b. As b varies, the range of m (b) is

(A) [0, 1] (B) (0, 1/2]

(C) $\left[\frac{1}{2}, 1\right]$ (D) (0, 1]

Solution: (D)

We have, $f(x) = (1 + b^2)$

$$\left[x^2 + \frac{2b}{1+b^2}x + \frac{b^2}{(1+b^2)^2}\right] - \frac{b^2}{1+b^2} + 1$$

$$= (1 + b^2)\left(x + \frac{b}{1+b^2}\right)^2 + \frac{1}{1+b^2} \geq \frac{1}{1+b^2}$$

$\therefore\ m\,(b) = \frac{1}{1+b^2}$. So, range of $m\,(b) = (0, 1]$.

26. If $f(x) = xe^{x\,(1-x)}$ then $f(x)$ is

(A) increasing on $\left[-\frac{1}{2}, 1\right]$

(B) decreasing on R

(C) increasing on R

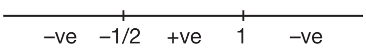
(D) decreasing on $\left[-\frac{1}{2}, 1\right]$

Solution: (A)

We have, $f'(x) = e^{x\,(1-x)} + x \cdot e^{x\,(1-x)} \cdot (1 - 2x)$

$= e^{x\,(1-x)}\,[1 + x\,(1 - 2x)] = e^{x\,(1-x)} \cdot (-2x^2 + x + 1)$.

Now, the sign-scheme for $-2x^2 + x + 1$ is



$\therefore f'(x) \geq 0$ if $x \in \left[\frac{-1}{2}, 1\right]$ because $e^{x\,(1-x)}$ is always positive.

So, $f(x)$ is increasing on $\left[-\frac{1}{2}, 1\right]$.

27. The function $f(x) = 2x^2 - \log|x|, x \neq 0$ is increasing in the interval

(A) $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

(B) $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

(C) $\left(-\frac{1}{2}, \frac{1}{2}\right)$

(D) None of these

Solution: (A)

We have, $f(x) = 2x^2 - \log|x|, x \neq 0$

Case I. When $x < 0, f(x) = 2x^2 - \log|x| = 2x^2 - \log(-x)$

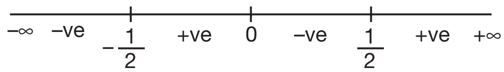
$\therefore\ f'(x) = 4x - \frac{1}{(-x)}(-1) = 4x - \frac{1}{x} = \frac{4x^2 - 1}{x}$

Case II. When $x > 0, f(x) = 2x^2 - \log|x| = 2x^2 - \log x$

$\therefore\ f'(x) = 4x - \frac{1}{x}$

Thus, when $x < 0$ or $x > 0, f'(x) = 4x - \frac{1}{x} = \frac{4x^2 - 1}{x}$

Sign scheme for $\frac{4x^2 - 1}{x}$:



Thus, $f(x)$ is an increasing function in the interval

$$\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right).$$

MAXIMA AND MINIMA OF FUNCTIONS

Local Maximum

A function $y = f(x)$ is said to have a local maximum value at a point $x = a$, if $f(x) \leq f(a), \forall\ x \in (a - h, a + h)$, for small $h > 0$, i.e. f(a) is the greatest of all the values of $f(x)$ in the interval $(a - h, a + h)$.

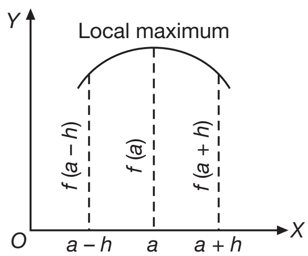


Fig. 14.9

The point $x = a$ is called a point of local maximum of the function $f(x)$.

Local Minimum

A function $y = f(x)$ is said to have a local minimum value at a point $x = a$, if $f(x) \geq f(a)$, $\forall\, x \in (a - h, a + h)$, for small $h > 0$, i.e., $f(a)$ is the smallest of all the values of $f(x)$ in the interval $(a - h, a + h)$.

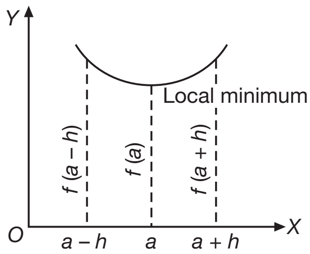


Fig. 14.10

The point $x = a$ is called a point of local minimum of the function $f(x)$.

REMEMBER

(i) The points at which a function attains either the local maximum value or local minimum value are called the *extreme points* and both local maximum and local minimum values are called the extreme values of the function $f(x)$.

(ii) The local maximum and local minimum values are also known as *relative maximum and relative minimum values* respectively.

METHOD TO DETERMINE THE POINTS OF LOCAL MAXIMA AND LOCAL MINIMA

Method I (First Derivative Test)

1. For the function $y = f(x)$, find $f'(x)$.
2. Put $f'(x) = 0$ and solve this equation for x. Let its roots be a, b, c etc. These points are called *stationary points* or *critical points*.
3. At $x = a$, determine the sign of $f'(x)$ for values of x slightly less than a and that for values of x slightly greater than a.
   (i) If $f'(x)$ changes sign from positive to negative as x increases through a, then $x = a$ is a point of maximum.

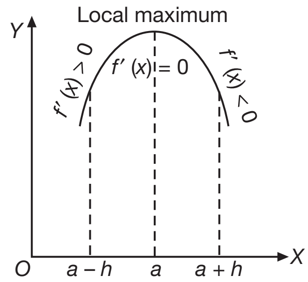


Fig. 14.11

   (ii) If $f'(x)$ changes sign from negative to positive as x increases through a, then $x = a$ is a point of minimum.

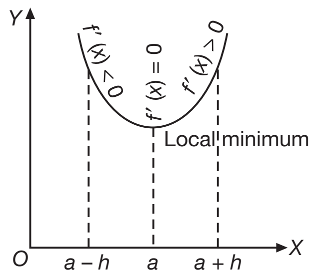


Fig. 14.12

   (iii) $f'(x)$ does not change sign as x increases through a, then $x = a$ is neither a point of maximum nor a point of minimum. Such a point is called a point of inflexion.

We repeat this process for other values of x and examine them for maxima or minima.

CAUTION

- A function may have maxima or minima at a point without being derivable at the point.
- If $f(a)$ does not exist, the there is no question of extrema at $x = a$.

Method II (Second Derivative Test)

1. For the function $y = f(x)$, find $f'(x)$ and $f''(x)$.
2. Put $f'(x) = 0$ and solve this equation for x. Let its roots be a, b, c etc.
3. At $x = a$
   (i) if $f''(\text{a}) < 0$, then $x = a$ is a point of local maxima;
   (ii) if $f''(\text{a}) > 0$, then $x = a$ is a point of local minima;
   (iii) if $f''(\text{a}) = 0$, we cannot say any thing.

GREATEST AND LEAST VALUES OF A FUNCTION IN A CLOSED INTERVAL (ABSOLUTE MAXIMUM AND ABSOLUTE MINIMUM)

If $f(x)$ is continuous in an interval $[a, b]$, then greatest or absolute maximum value of $f(x) = \max. [f(a), f(b)$, values of $f(x)$ at all critical points in $(a, b)]$.

Also, least or absolute minimum value of $f(x) = \min. [f(a), f(b)$, values of $f(x)$ at all critical points in $(a, b)]$.

IMPORTANT POINTS

If a function is defined and continuous on an interval which is not a closed interval, then it cannot have any greatest or least value other than local maximum or local minimum values.

SOLVED EXAMPLES

28. If $P(x) = a_0 + a_1x^2 + a_2x^4 + ... + a_nx^{2n}$ be a polynomial in $x \in R$ with $0 < a_1 < a_2 ... < a_n$, then $P(x)$ has

(A) no point of minimum
(B) only one point of minimum
(C) only two points of minimum
(D) None of these

Solution: (B)

We have,

$$P(x) = a_0 + a_1x^2 + a_2x^4 + ... + a_nx^{2n}$$

$$\Rightarrow \quad P'(x) = 2a_1x + 4a_2x^3 + ... + 2na_nx^{2n-1}$$

For maximum or minimum, $P'(x) = 0$

$$\Rightarrow \quad x(2a_1 + 4a_2x^2 + ... + 2na_nx^{2n-2}) = 0$$

$\Rightarrow \quad x = 0$ (each $a_i > 0$ and powers of x are even)

Now, $P''(x) = 2a_1 + 12a_2x^2 + ... + 2n(2n-1)a_nx^{2n-2}$

$\therefore \quad P''(x)]_{x=0} = 2a_1 > 0$ i.e. $P(x)$ has a minimum at $x = 0$ only.

29. The fraction exceeding its pth power by the greatest number possible, where $p \geq 2$, is

(A) $\left(\frac{1}{p}\right)^{1/p-1}$
(B) $\left(\frac{1}{p}\right)^{p-1}$
(C) $p^{1/p-1}$
(D) None of these

Solution: (A)

Let $y = x - x^p$, where x is the fraction

$$\Rightarrow \quad \frac{dy}{dx} = 1 - px^{p-1}$$

For maximum or minimum, $\frac{dy}{dx} = 0$

$$\Rightarrow \quad 1 - px^{p-1} = 0 \Rightarrow x = \left(\frac{1}{p}\right)^{1/(p-1)}.$$

Now, $$\frac{d^2y}{dx^2} = -p(p-1)x^{p-2}$$

$$\therefore \quad \left.\frac{d^2y}{dx^2}\right]_{x=\left(\frac{1}{p}\right)^{1/(p-1)}} = -p(p-1)\left(\frac{1}{p}\right)^{p-2/(p-1)} < 0$$

$$\therefore \quad y \text{ is maximum at } x = \left(\frac{1}{p}\right)^{1/(p-1)}$$

30. The shortest distance of the point (0, 0) from the curve $y = \frac{1}{2}(e^x + e^{-x})$ is

(A) 2
(B) 1
(C) 3
(D) None of these

Solution: (B)

Let $P(x, y)$ be the point on the curve which is nearest to $O(0, 0)$.

$$\text{Let } z = OP^2 = x^2 + y^2 = x^2 + \frac{1}{4}(e^{2x} + e^{-2x} + 2).$$

$$\Rightarrow \quad \frac{dz}{dx} = 2x + \frac{1}{2}(e^{2x} - e^{-2x})$$

For maximum or minimum, $\frac{dz}{dx} = 0$

$$\Rightarrow \quad 2x + \frac{1}{2}(e^{2x} - e^{-2x}) = 0$$

$$\Rightarrow \quad \frac{e^{-2x} - e^{2x}}{2} = 2x$$

$\Rightarrow \quad x = 0$ is a solution and then $y = \frac{1}{2}(e^0 + e^0) = 1.$

Also, $\frac{d^2y}{dx^2} = 2 + e^{2x} + e^{-2x} > 0,$

hence, z is minimum.

$\therefore \quad$ The shortest distance $OP = \sqrt{0^2 + 1^2} = 1.$

31. Let the function $f(x)$ be defined as

$$f(x) = \begin{cases} \tan^{-1}\alpha - 3x^2, & 0 < x < 1 \\ -6x, & x \geq 1 \end{cases}$$

$f(x)$ can have a maximum at $x = 1$ if the value of α is

(A) 0 (B) 2
(C) 1 (D) None of these

Solution: (D)

We have,

$$f'(x) = \begin{cases} -6x, & 0 < x < 1 \\ -6, & x \geq 1 \end{cases}$$

$\therefore \quad f'(1-h) = -6(1-h) < 0$

and $\quad f'(1+h) = -6 < 0$

Since $f'(x)$ does not change sign as x passes through 1, therefore, $f(x)$ does not have a maximum or minimum at $x = 1$, whatever be the value of α.

32. If the roots of the equation $x^3 - ax^2 + 4x - 8 = 0$ are real and positive, then the minimum value of a is

(A) 2 (B) 6
(C) $3\sqrt[3]{4}$ (D) None of these

Solution: (B)

Let α, β, γ be the roots of the given equation.

Then, $\alpha + \beta + \gamma = a$, $\alpha\beta + \alpha\gamma + \beta\gamma = 4$, $\alpha\beta\gamma = 8$.

Since AM $\geq$ GM $\Rightarrow \dfrac{\alpha+\beta+\gamma}{3} \geq \sqrt[3]{\alpha\beta\gamma}$

$\Rightarrow \quad \dfrac{a}{3} \geq \sqrt[3]{8} \Rightarrow a \geq 6$

$\therefore$ The minimum value of $a = 6$

33. For the function $f(x) = \displaystyle\int_0^x \frac{\sin t}{t}\,dt$, where $x > 0$,

(A) maximum occurs at $x = n\pi$, n even
(B) minimum occurs at $x = n\pi$, n odd
(C) maximum occurs at $x = n\pi$, n odd
(D) minimum occurs at $x = n\pi$, n even

Solution: (C, D)

We have,

$$f'(x) = \frac{\sin x}{x} \text{ and } f''(x) = \frac{x\cos x - \sin x}{x^2}$$

For maximum or minimum, $f'(x) = 0$

$\Rightarrow \quad \dfrac{\sin x}{x} = 0 \Rightarrow \sin x = 0;\ x \neq 0.$

$\therefore \quad x = n\pi;\ n = 1, 2, 3, \ldots\ (\because\ x > 0)$

At $\quad x = n\pi, f''(x) = \dfrac{n\pi\cos n\pi - \sin n\pi}{(n\pi)^2} = \dfrac{\cos n\pi}{n\pi}$

$$= \frac{(-1)^n}{n\pi}$$

$\therefore$ Extreme points are $x = n\pi$, $n = 1, 2, 3, \ldots$, where the maximum occurs at $x = \pi, 3\pi, 5\pi, \ldots$ and the minimum occurs at $x = 2\pi, 4\pi, 6\pi, \ldots$

34. The difference between the greatest and the least value of the function

$$f(x) = \int_0^x (6t^2 - 24)\,dt \text{ on } [1, 3] \text{ is}$$

(A) 14 (B) 10
(C) 4 (D) None of these

Solution: (A)

We have, $f(x) = \displaystyle\int_0^x (6t^2 - 24)\,dt$

$\Rightarrow \quad f'(x) = (6x^2 - 24) \times 1$

$\quad f'(x) = 0 \Rightarrow 6x^2 - 24 = 0 \Rightarrow x = \pm 2.$

But $x = -2 \notin [1, 3]$. So $x = 2$ is the only critical point.

Now, $f(1) = \displaystyle\int_0^1 (6t^2 - 24)\,dt = \left(\frac{6t^3}{3} - 24t\right)\Bigg|_0^1 = -22,$

$f(2) = \displaystyle\int_0^2 (6t^2 - 24)\,dt = \left(\frac{6t^3}{3} - 24t\right)\Bigg|_0^2 = -32$

and $f(3) = \displaystyle\int_0^3 (6t^2 - 24)\,dt = \left(\frac{6t^3}{3} - 24t\right)\Bigg|_0^3 = -18.$

Hence, the greatest value of $f(x)$ is -18 which it attains at $x = 3$ and the least value is -32 which is attained at $x = 2$.

Thus, the difference between the greatest and the least value of the function

$$= f(3) - f(2) = -18 + 32 = 14$$

35. The minimum value of $e^{(x^4 - x^3 + x^2)}$ is

(A) e (B) e^2
(C) 1 (D) None of these

Solution: (C)

$e^{(x^4 - x^3 + x^2)}$ is minimum when $(x^4 - x^3 + x^2)$ is minimum.

Since $x^4 - x^3 + x^2 = x^2(x^2 - x + 1)$

$$= x^2\left[\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}\right]$$

$$\geq 0, \forall x \in R$$

$\therefore$ The minimum value of $(x^4 - x^3 + x^2)$ is 0 for $x = 0$.

Hence, the minimum value of $e^{(x^4 - x^3 + x)}$ is $e^0 = 1$.

36. The minimum value of the function
$f(x) = 2\,|x - 2| + 5\,|x - 3|, \forall x \in R$ is
(A) 3 (B) 2
(C) 5 (D) 7

Solution: (B)
We have,
$$f(x) = 2\,|x - 2| + 5\,|x - 3|$$
$$= 2(2 - x) + 5(3 - x) = 19 - 7x, \text{ if } x < 2$$
$$= 5, \text{ if } x = 2$$
$$= 2(x - 2) + 5(3 - x) = 11 - 3x, \text{ if } 2 < x < 3$$
$$= 2(3 - 2) = 2, \text{ if } x = 3$$
$$= 2(x - 2) + 5(x - 3) = 7x - 19, \text{ if } x > 3$$

Thus, we find that $f(x)$ has a minimum value 2 at $x = 3$.

37. If $(x - a)^{2n}(x - b)^{2m+1}$, where m and n are positive integers and $a > b$, is the derivative of a function f, then
(A) $x = a$ gives neither a maximum nor a minimum
(B) $x = a$ gives a maximum
(C) $x = b$ gives a minimum
(D) $x = b$ gives neither a maximum nor a minimum

Solution: (A, C)

Let $f'(x) = (x - a)^{2n}(x - b)^{2m+1}$

The extreme values of f are given by $f'(x) = 0$

$$\Rightarrow (x - a)^{2n} \cdot (x - b)^{2m+1} = 0 \Rightarrow x = a, b.$$

For $x < b$, $(x - b)^{2m+1}$ is negative and for $x > b$, $(x - b)^{2m+1}$ is positive ($2m + 1$ is odd).

Thus, f' changes sign from negative to positive as x passes through b and so, f has a minimum at $x = b$.

Since $2n$ is an even integer, $(x - a)^{2n}$ does not change sign as x passes through a i.e., $f'(x)$ does not change sign as x passes through a. Hence, f has neither a maximum nor a minimum at $x = a$.

38. The value of k so that the sum of the cubes of the roots of the equation $x^2 - kx + (2k - 3) = 0$ assumes the minimum value, is
(A) $k = 1$ (B) $k = 3$
(C) $k = 0$ (D) None of these

Solution: (B)
Let α and β be the roots of the given equation. Then $\alpha + \beta = k$ and $\alpha\beta = (2k - 3)$

Let $z = \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
$$= k^3 - 3k(2k - 3) = k^3 - 6k^2 + 9k$$

$$\therefore \quad \frac{dz}{dk} = 3k^2 - 12k + 9 = 3(k^2 - 4k + 3)$$
$$= 3(k - 1)(k - 3)$$

Now, $\frac{dz}{dk} = 0 \Rightarrow 3(k - 1)(k - 3) = 0 \Rightarrow k = 1, 3$

Also, $\left.\frac{d^2z}{dk^2}\right]_{k=3} = (6k - 12)\big|_{k=3} = 6(3) - 12 > 0$

Hence, z is minimum when $k = 3$.

39. If $f(x) = |x| + |x - 1| + |x - 2|$, then
(A) $f(x)$ has minima at $x = 1$
(B) $f(x)$ has maxima at $x = 0$
(C) $f(x)$ has neither maxima nor minima at $x = 0$
(D) $f(x)$ has neither maxima nor minima at $x = 2$.

Solution: (A, C, D)
We have,

$$f(x) = |x| + |x - 1| + |x - 2|$$

$$= \begin{cases} -3x + 3, & x < 0 \\ -x + 3, & 0 \leq x < 1 \\ x + 1, & 1 \leq x < 2 \\ 3x - 3, & x \geq 2 \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} -3 & x < 0 \\ \text{does not exist} & x = 0 \\ -1 & 0 < x < 1 \\ \text{does not exist} & x = 1 \\ 1 & 1 < x < 2 \\ \text{does not exist} & x = 2 \\ 3 & x > 2 \end{cases}$$

Clearly, $f(x)$ has minima at $x = 1$ and neither maxima nor minima at $x = 0$ and $x = 2$.

40. The function

$$f(x) = \int_{-1}^{x} t(e^t - 1)(t - 1)(t - 2)^3(t - 3)^5\,dt$$

has a local minimum at $x =$
(A) 0 (B) 1
(C) 2 (D) 3

Solution: (B, D)

We have,

$$f'(x) = x(e^x - 1)(x - 1)(x - 2)^3 (x - 3)^5$$

$$\therefore \quad f'(x) = 0 \Rightarrow x = 0, 1, 2, 3$$

Sign scheme for $f'(x)$:

–∞ –ve 0 –ve 1 +ve 2 –ve 3 +ve ∞

Clearly, $f(x)$ has local minimum at $x = 1$ and $x = 3$.

41. Let $f(x) = \begin{cases} |x|, & 0 < |x| \le 2 \\ 1, & x = 0 \end{cases}$. Then at $x = 0$, f has

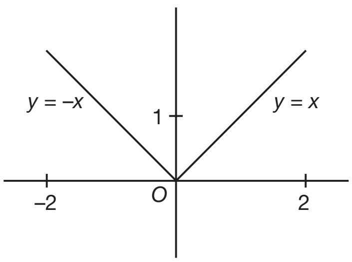
(A) a local maximum (B) no local maximum
(C) a local minimum (D) no extremum

Solution: (A)

We have,

$$f(x) = \begin{cases} -x, & -2 \le x < 0 \\ 1, & x = 0 \\ x, & 0 < x \le 2 \end{cases}$$

The graph of $f(x)$ is



Clearly, from the graph,

$$f(0) = 1, f(0 - \varepsilon) < 1,$$
$$f(0 + \varepsilon) < 1$$

where ε is small and positive.

$\therefore$ $f(x)$ has a local maximum at $x = 0$.

42. The number of values of x where the function $f(x) = \cos x + \cos(\sqrt{2}\, x)$ attains its maximum is

(A) 0 (B) 1
(C) 2 (D) infinite

Solution: (B)

We have, $f(x) = \cos x + \cos(\sqrt{2}\, x)$

$$\Rightarrow \quad |f(x)| = \left|\cos x + \cos\sqrt{2}x\right| \le |\cos x| + \left|\cos\sqrt{2}x\right|$$

$$= 1 + 1 = 2, \forall x \in R.$$

$\therefore$ Maximum value of $f(x) = 2$.

This requires $\cos x = 1$ and $\cos\sqrt{2}\,x = 1$

$$\Rightarrow \quad x = 2n\pi \text{ and } \sqrt{2}\,x = 2m\pi,\ n, m \in I.$$

$$\Rightarrow \quad 2n\pi = \frac{2m\pi}{\sqrt{2}} \Rightarrow n = \frac{m}{\sqrt{2}}$$

This is possible only when $n = m = 0$.

$\therefore$ There is only one value ($x = 0$) at which $f(x)$ attains its maximum value.

43. The minimum value of $27^{\cos 2x} \cdot 81^{\sin 2x}$ is

(A) $\frac{1}{243}$ (B) –5
(C) $\frac{1}{5}$ (D) None of these

Solution: (A)

Let $y = 27^{\cos 2x} \cdot 81^{\sin 2x} = 3^{3\cos 2x + 4\sin 2x}$

y will be minimum when $3\cos 2x + 4\sin 2x$ is minimum.

Let $Z = 3\cos 2x + 4\sin 2x$

Put $3 = r\cos\theta$, $4 = r\sin\theta$

Then, $r = \sqrt{3^2 + 4^2} = 5$ and $\tan\theta = \frac{4}{3}$

i.e., $\theta = \tan^{-1}\left(\frac{4}{3}\right)$

$$\therefore \quad Z = 5\cos(2x - \theta) \Rightarrow -5 \le Z \le 5$$

$$\therefore \quad \text{Min. } Z = -5 \Rightarrow \text{Min. } y = 3^{-5} = \frac{1}{243}$$

44. If $h(x) = f(x) + f(-x)$, then $h(x)$ has got an extreme value at a point where $f'(x)$ is

(A) even function (B) odd function
(C) zero (D) None of these

Solution: (A)

We have, $h'(x) = f'(x) - f'(-x)$

For extreme values of $h(x)$, $h'(x) = 0$

$$\Rightarrow \quad f'(x) = f'(-x) \Rightarrow f'(x) \text{ is an even function.}$$

45. Let $f(x) = \int_0^x \frac{\cos t}{t}\, dt$ $(x > 0)$; then for $x = (2n + 1)\frac{\pi}{2}$, $f(x)$ has

(A) minima when $n = 0, 2, 4, \ldots$
(B) maxima when $n = 0, 2, 4, 6, \ldots$
(C) neither max. nor min. when $n = -1, -3, -5, \ldots$
(D) None of these

Solution: (B)

We have, $f'(x) = \dfrac{\cos x}{x}$.

$\therefore \quad f'(x) = 0 \Rightarrow \cos x = 0 \Rightarrow x = (2n+1)\dfrac{\pi}{2}, n \in I.$

Also, $f''(x) = \dfrac{-x\sin x - \cos x}{x^2}$.

$$\therefore \quad f''(x)\big]_{x=(2n+1)\frac{\pi}{2}} = \frac{-(2n+1)\frac{\pi}{2}\sin(2n+1)\frac{\pi}{2} - 0}{\left[(2n+1)\frac{\pi}{2}\right]^2}$$

$$= \frac{-2(-1)^n}{(2n+1)\pi}.$$

< 0, for $n = 0, 2, 4, 6, ...$

$\therefore \quad f(x)$ has maxima when $n = 0, 2, 4, 6, \ldots$

46. The minimum value of $2^{(x^2-3)^3+27}$ is

(A) 1 (B) 2
(C) 2^{27} (D) None of these

Solution: (A)

$2^{(x^2-3)^3+27}$ is minimum when $(x^2 - 3)^3 + 27$ is minimum.

$$\begin{aligned}\text{Since } (x^2-3)^3 + 27 &= x^6 - 9x^4 + 27x^2 \\ &= x^2(x^4 - 9x^2 + 27) \\ &= x^2\left[\left(x^2 - \frac{9}{2}\right)^2 + \frac{27}{4}\right] \\ &\geq 0, \text{ for all } x,\end{aligned}$$

$\therefore \quad$ minimum value of $(x^2 - 3)^3 + 27 = 0$.

Thus, minimum value of $2^{(x^2-3)^3+27}$ is $2^0 = 1$

47. $f(x) = 1 + [\cos x]\, x$, in $0 < x \leq \dfrac{\pi}{2}$

(A) has a minimum value 0
(B) has a maximum value 2
(C) is continuous in $\left[0, \dfrac{\pi}{2}\right]$
(D) is not differentiable at $x = \dfrac{\pi}{2}$

Solution: (C)

Since $f(x) = 1$ in $0 < x < \dfrac{\pi}{2}$ (as $[\cos x] = 0$)

$\therefore \quad f(x)$ is continuous in $\left[0, \dfrac{\pi}{2}\right]$

48. For all real x, the minimum value of $\dfrac{1-x+x^2}{1+x+x^2}$ is

(A) 0 (B) $\dfrac{1}{3}$
(C) 1 (D) 3

Solution: (B)

Let $Z = \dfrac{1-x+x^2}{1+x+x^2}$

$\Rightarrow \quad Z + Zx + Zx^2 = 1 - x + x^2$

$\Rightarrow \quad Zx^2 - x^2 + Zx + x + Z - 1 = 0$

$\Rightarrow \quad x^2(Z-1) + x(Z+1) + (Z-1) = 0$

For x to the real, $B^2 - 4AC \geq 0$

$\Rightarrow \quad (Z+1)^2 - 4(Z-1)(Z-1) \geq 0$

$\Rightarrow \quad Z^2 + 2Z + 1 - 4Z^2 + 8Z - 4 \geq 0$

$\Rightarrow \quad -3Z^2 + 10Z - 3 \geq 0$ i.e. $3Z^2 - 10Z + 3 \leq 0$

$\Rightarrow \quad (Z-3)(3Z-1) \leq 0 \Rightarrow \dfrac{1}{3} \leq Z \leq 3$

Therefore, minimum value of $Z = \dfrac{1}{3}$.

49. If $\log_{10}(x^3 + y^3) - \log_{10}(x^2 + y^2 - xy) \leq 2$ then the maximum value of xy is

(A) 2500 (B) 3000
(C) 1200 (D) 3500

Solution: (A)

$\log_{10} \dfrac{(x^3 + y^3)}{x^2 + y^2 - xy} \leq 2$ and $x + y > 0$

$\Rightarrow \quad 0 < x + y \leq 100$

$\Rightarrow \quad$ maximum value of $xy = 2{,}500$.

CONCAVITY AND CONVEXITY OF A FUNCTION

Concavity of Function

If $f''(x) > 0$ in the interval (a, b), then shape of $f(x)$ in interval (a, b) is concave when observed from upwards (i.e., concave upwards) or convex down.

Geometrically, a curve is concave upward in the interval $[a, b]$ if all points on the curve lie above the tangent to the curve at any point in the interval $[a, b]$.

DO NOT FORGET

If the curve is concave upward, then the slope of the tangent increases as x increases i.e., $f'(x)$ is strictly increasing in $[a, b]$

Convexity of Function

If $f''(x) < 0$ in the interval (a, b), then shape of $f(x)$ in interval (a, b) is convex when observed from upwards (i.e., convex upwards) or concave down.

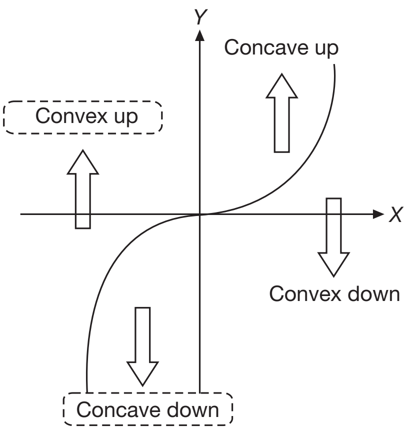


Fig. 14.13

Geometrically, a curve is concave downward in the interval $[a, b]$ if all points on the curve lie above the tangent to the curve at any point in the interval $[a, b]$.

DO NOT FORGET

If the curve is concave downward, then the slope of the tangent decreases as x increases i.e., $f'(x)$ is strictly decreasing in $[a, b]$.

$\Rightarrow f''(x) < 0 \ \forall \ x \in [a, b]$

POINT OF INFLEXION

If at $x = a$, the shape of the curve changes from concave to convex or from convex to concave, then $x = a$ is known as the point of inflexion.

Method to Evaluate Point of Inflexion

Points of inflexion can be obtained by equating $f''(x) = 0$. It is not necessary that all values of x which are obtained by equating $f''(x) = 0$ are points of inflexion. Only those values of x for whcih $f''(x)$ changes sign are points of inflexion.

Higher Order Derivative Test to Determine Local maxima, Local Minima and Point of Inflexion

Let $f(x)$ be a differentiable function in an interval I and let $x = a$ be a point lying in the interior of I such that

1. $f'(a) = f''(a) = f'''(a) = \ldots = f^{n-1}(a) = 0$ and
2. $f^n(a)$ exists and is non-zero, then:
   - **(a)** If n is even and $f^n(a) < 0$, then $x = a$ is a point of local maximum.
   - **(b)** If n is even and $f^n(a) > 0$, then $x = a$ is a point of local minimum.
   - **(c)** If n is odd and $f^n(a) > 0$, then $x = a$ is a point of inflexion where shape of curve changes from convex upwards to concave upwards.
   - **(d)** If n is odd and $f^n(a) < 0$, then $x = a$ is a point of inflexion, where shape of curve changes from concave upwards to convex upwards.

IMPORTANT POINTS

This test is used only when second derivative fails to decide between local maximum and local minimum.

i.e., when at $x = a$, $f'(a) = 0$ and $f''(a) = 0$.

TRICK(S) FOR PROBLEM SOLVING

- Maxima and minima occur alternately,that is between two maxima there is one minimum and vice-versa.
- If $f(x) \to \infty$ as $x \to a$ or b and $f'(x) = 0$ only for one value of x (say c) between a and b, then $f(c)$ is necessarily the minimum and the least value.

  If $f(x) \to -\infty$ as $x \to a$ or b, then $f(c)$ is necessarily the maximum and the greatest value.
- If a function is strictly increasing in $[a, b]$, then $f(a)$ is local minimum and $f(b)$ is local maximum.
- If a function is strictly decreasing in $[a, b]$, then $f(a)$ is local maximum and $f(b)$ is local minimum.

ROLLE'S AND LAGRANGE'S MEAN VALUE THEOREM

Rolle's Theorem

If a function f defined on the closed interval $[a, b]$, is

1. continuous on $[a, b]$,
2. derivable on (a, b), and
3. $f(a) = f(b)$, then there exists atleast one real number c between a and b $(a < c < b)$ such that $f'(c) = 0$.

Geometrical Interpretation

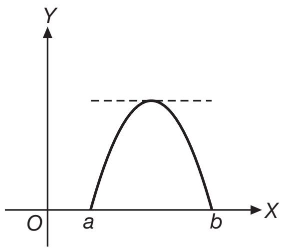


Fig. 14.14

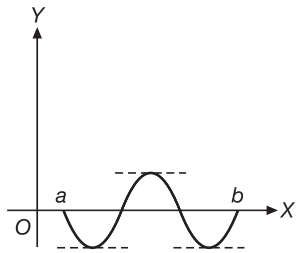


Fig. 14.15

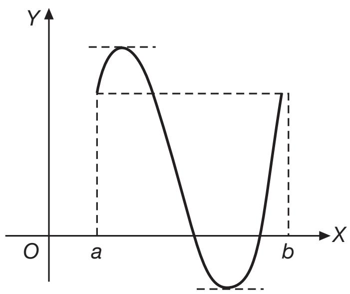


Fig. 14.16

The conclusion is that there is a point c between a and b such that the tangent to the graph at $[c, f(c)]$ is parallel to the x-axis.

Algebraic Interpretation

Between two zeros a and b of $f(x)$ (i.e., between two roots a and b of $f(x) = 0$) there exists atleast one zero of $f'(x)$.

TRICK(S) FOR PROBLEM SOLVING

Suppose a and b are two real numbers such that

- $f(x)$ and its derivative $f'(x)$ are continuous for $a \le x \le b$.
- $f(a)$ and $f(b)$ have opposite signs.
- $f'(x)$ is different from zero for all values of x between a and b.

Then, there is one and only real root of the equation $f(x) = 0$ between a and b.

SOLVED EXAMPLES

50. If the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0,$$

n positive integer, has two different real roots α and β, then between α and β, the equation

$$n a_n x^{n-1} + (n-1)\, a_{n-1} x^{n-2} + \ldots + a_1 = 0 \text{ has}$$

(A) exactly one root (B) atmost one root
(C) atleast one root (D) no root

Solution: (C)

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, which is a polynomial function in x of degree n. Hence $f(x)$ is continuous and differentiable for all x.

Let $\alpha < \beta$. We are given, $f(\alpha) = 0 = f(\beta)$.

By Rolle's theorem, $f'(c) = 0$ for some value c,

$$\alpha < c < \beta$$

Hence the equation

$$f'(x) = n a_n x^{n-1} + (n-1)\, a_{n-1} x^{n-2} + \ldots + a_1 = 0$$

has atleast one root between α and β.

51. If $a + b + c = 0$, then the equation $3ax^2 + 2bx + c = 0$ has, in the interval (0, 1)

(A) atleast one root (B) atmost one root
(C) no root (D) None of these

Solution: (B)

Let $f(x) = ax^3 + bx^2 + cx, x \in [0, 1]$.

$\therefore \quad f'(x) = 3ax^2 + 2bx + c.$

Since $f(x)$ is a polynomial function of x, it is continuous and differentiable for all $x \in [0, 1]$.

Also, $f(0) = 0; f(1) = a + b + c = 0.$

$\therefore \quad f(0) = f(1)$

Applying Rolle's theorem, $f'(k) = 0$ for atleast one value k, $0 < k < 1$. Hence k is a root of the equation

$$3ax^2 + 2bx + c = 0, \text{ where } 0 < k < 1$$

52. If the equation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x = 0$ has a positive root $x = \alpha$, then the equation $n a_n x^{n-1} + (n-1)$ $a_{n-1} x^{n-2} + \ldots + a_1 = 0$ has a positive root, which is

(A) smaller than α
(B) greater than α
(C) equal to α
(D) greater than or equal to α

Solution: (A)

$$\text{Let } f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x.$$

Then $f(\alpha) = 0$ (Given). Also $f(0) = 0$.

Moreover, $f(x)$ is continuous and differentiable in $[0, \alpha]$ as it is a polynomial function of x. Hence, by Rolle's theorem, there exists a c in $(0, \alpha)$ such that $f'(x) = 0$ for $x = c$ i.e.

$$na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + 2a_2 x + a_1 = 0.$$

53. If $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots + \frac{a_{n-1}}{2} + a_n = 0$, then the equation $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$ has, in the interval (0, 1),

(A) exactly one root (B) atleast one root
(C) atmost one root (D) no root

Solution: (B)

Let

$$f(x) = a_0\frac{x^{n+1}}{n+1} + a_1\frac{x^n}{n} + a_2\frac{x^{n-1}}{n-1} + \ldots + a_{n-1}\frac{x^2}{2} + a_n x$$

Then $f(x)$ is continuous and differentiable in [0, 1], as it is a polynomial function of x.

Also, $f(0) = 0$

and $f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots + \frac{a_{n-1}}{2} + a_n = 0.$ (Given)

Hence, by Rolle's theorem, there exists atleast one real number $c \in (0, 1)$ such that $f'(c) = 0$ i.e., c is a root of the equation $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$.

54. The equation $x \log x = 3 - x$ has, in the interval (1, 3),

(A) exactly one root (B) atmost one root
(C) atleast one root (D) no root

Solution: (C)

Let $f(x) = (x - 3) \log x$

Then, $f(1) = -2 \log 1 = 0$ and $f(3) = (3 - 3) \log 3 = 0$.

As, $(x - 3)$ and $\log x$ are continuous and differentiable in [1, 3], therefore $(x - 3) \log x = f(x)$ is also continuous and differentiable in [1, 3]. Hence, by Rolle's theorem, there exists a value of x in (1, 3) such that

$$f'(x) = 0$$

$$\Rightarrow \quad \log x + (x-3)\frac{1}{x} = 0$$

$$\Rightarrow \quad x \log x = 3 - x.$$

55. Between any two real roots of the equation $e^x \sin x = 1$, the equation $e^x \cos x = -1$ has

(A) atleast one root (B) exactly one root
(C) atmost one root (D) no root

Solution: (A)

Let $\alpha, \beta \, (\alpha < \beta)$ be any two real roots of

$$f(x) = e^{-x} - \sin x.$$

Then, $f(\alpha) = 0 = f(\beta)$

Moreover, $f(x)$ is continuous and differentiable for $x \in [\alpha, \beta]$.

Hence, from Rolle's theorem, there exists atleast one x in (α, β) such that

$$f'(x) = 0 \Rightarrow -e^{-x} - \cos x = 0$$
$$\Rightarrow -e^{-x}(1 + e^x \cos x) = 0 \Rightarrow e^x \cos x = -1.$$

56. If a, b, c be non-zero real numbers such that

$$\int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx = \int_0^2 (1+\cos^8 x)(ax^2+bx+c)\,dx = 0,$$

then the equation $ax^2 + bx + c = 0$ will have

(A) one root between 0 and 1 and other root between 1 and 2
(B) both the roots between 0 and 1
(C) both the roots between 1 and 2
(D) None of these

Solution: (A)

$$\text{Let } f(y) = \int_0^y (1+\cos^8 x)(ax^2+bx+c)\,dx$$

$$\Rightarrow \quad f'(y) = (1 + \cos^8 y)(ay^2 + by + c) \qquad (1)$$

Now, $f(1) = \int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx = 0$

and $f(2) = \int_0^2 (1+\cos^8 x)(ax^2+bx+c)\,dx = 0$

Also, $f(0) = 0$

$\therefore \quad f(0) = f(1) = f(2)$

Now by Rolle's theorem for $f(x)$ in [0, 1].

$f'(\alpha) = 0$, for atleast one α, $0 < \alpha < 1$

and by Rolle's theorem for $f(x)$ in [1, 2],

$f'(\beta) = 0$, for atleast one β, $1 < \beta < 2$.

From (1),

$$f'(\alpha) = 0 \Rightarrow (1 + \cos^8\alpha)(a\alpha^2 + b\alpha + 2) = 0.$$

But $1 + \cos^8\alpha \neq 0$,

$\therefore \quad a\alpha^2 + b\alpha + c = 0$, i.e., α is a root of the equation $ax^2 + bx + c = 0$.

Similarly $f'(\beta) = 0 \Rightarrow a\beta^2 + b\beta + c = 0$,

i.e., β is a root of the equation $ax^2 + bx + c = 0$.

But the equation $ax^2 + bx + c = 0$, being a quadratic equation, cannot have more than two roots.

$\therefore$ The equation $ax^2 + bx + c = 0$ has one root α between 0 and 1 and other root β between 1 and 2.

57. If $27a + 9b + 3c + d = 0$, then the equation $4ax^3 + 3bx^2 + 2cx + d = 0$ has atleast one real root lying between

(A) 0 and 1 (B) 1 and 3
(C) 0 and 3 (D) None of these

Solution: (C)

Let $f(x) = ax^4 + bx^3 + cx^2 + dx$

Then,

$$f(0) = 0 \text{ and } f(3) = 81a + 27b + 9c + 3d$$
$$= 3\,(27a + 9b + 3c + d)$$
$$= 0 \qquad (\because\ 27a + 9b + 3c + d = 0)$$

Therefore 0 and 3 are roots of the polynomial $f(x)$. So, by Rolle's theorem, there exists atleast one root of the polynomial $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$ lying between 0 and 3.

LAGRANGE'S MEAN VALUE THEOREM

If a function f defined on the closed interval $[a, b]$, is

1. continuous on $[a, b]$
2. derivable on (a, b), then there exists atleast one real number c between a and b $(a < c < b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrical Interpretation

The theorem states that between two points A and B on the graph of f there exists atleast one point where the tangent is parallel to the chord AB.

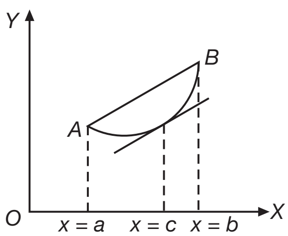


Fig. 14.17

SOLVED EXAMPLES

58. If $f(x)$ is differentiable in the interval [2, 5], where $f(2) = \frac{1}{5}$ and $f(5) = \frac{1}{2}$, then there exists a number c, $2 < c < 5$ for which $f'(c) =$

(A) $\frac{1}{2}$ (B) $\frac{1}{5}$
(C) $\frac{1}{10}$ (D) None of these

Solution: (C)

As $f(x)$ is differentiable in [2, 5], therefore, it is also continuous in [2, 5]. Hence, by mean value theorem, there exists a real number c in (2, 5) such that

$$f'(c) = \frac{f(5) - f(2)}{5 - 2} \Rightarrow f'(c) = \frac{\frac{1}{2} - \frac{1}{5}}{3} = \frac{1}{10}$$

59. The value of c in Lagrange's theorem for the function

$$f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$ in the interval [– 1, 1] is

(A) 0 (B) $\frac{1}{2}$
(C) $-\frac{1}{2}$ (D) non-existent in the interval

Solution: (D)

$f(x)$ is continuous in the interval [– 1, 1], but $f(x)$ is not differentiable at $x = 0$. Hence mean value theorem is not applicable. So, no c can be found.

60. Let f be a function which is continuous and differentiable for all real x. If $f(2) = -4$ and $f'(x) \geq 6$ for all $x \in [2, 4]$, then

(A) $f(4) < 8$ (B) $f(4) \geq 8$
(C) $f(4) \geq 12$ (D) None of these

Solution: (B)

By mean value theorem, there exists a real number $c \in (2, 4)$ such that

$$f'(c) = \frac{f(4) - f(2)}{4 - 2} \Rightarrow f'(c) = \frac{f(4) + 4}{2}$$

Since $f'(x) \geq 6\ \forall\ x \in [2, 4]$

$$\therefore\quad f'(c) \geq 6 \Rightarrow \frac{f(4) + 4}{2} \geq 6 \Rightarrow f(4) + 4 \geq 12$$

$$\Rightarrow \qquad f(4) \geq 8$$

APPLICATION OF $\dfrac{dy}{dx}$ AS A RATE MEASURE

The importance of the derivable functions in various practical problems of day to day life rests on the fact that the derivatives give us a measure of the rate of change of a function with respect to tis independent variable.

Let $y = f(x)$ be the given function. Take a fixed value x_1 for x and the corresponding value of y for $x = x_1$ is say y_1, i.e., $y_1 = f(x_1)$. Take another value x_2 of x and the corresponding value of y be y_2.

Let δx and δy denote the increments in x and y respectively. The increment ratio $\dfrac{\delta y}{\delta x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ is called the average rate of change with respect to x in the interval $[x_1, x_2]$. If we continue choosing the values of x in such a way that the interval $[x_1, x_2]$ shrinks to zero, i.e., $dx \to 0$ then according to the definition of derivative.

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_1) = \left(\frac{dy}{dx}\right)_{x = x_1}$$

And this is known as the rate of change of y with respect to x for the same value $x = x_1$.

Hence $\dfrac{dy}{dx}$ represents the actual rate of increases in y per unit increase in x for the particular value of x or $\dfrac{dy}{dx}$ is the rate at which y is changing with respect to x.

$$\text{Again, } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

In order words, if the rate of change of variables x and y are taken relative to time t, we have

$$\text{Rate of change of } y = \frac{dy}{dx} \times \text{rate of change of } x.$$

EXERCISES

Single Option Correct Type

1. The set of values of x for which $\log(1 + x) < x$, is

(A) $x < 0$ (B) $x > 0$
(C) $0 < x < 1$ (D) None of these

2. Let $f(x) = \cos 2\pi x + x - [x]$, where $[\cdot]$ denotes the greatest integer function. Then the number of points in $[0, 10]$ at which $f(x)$ assumes its local maximum value, is

(A) 10 (B) 9
(C) 0 (D) infinite

3. The function $f(x) = \dfrac{\sin x}{x}$ is decreasing in the interval

(A) $\left(-\dfrac{\pi}{2}, 0\right)$ (B) $\left(0, \dfrac{\pi}{2}\right)$
(C) $(0, \pi)$ (D) None of these

4. If $ax + \dfrac{b}{x} \geq c$ for all positive x, where $a, b > 0$, then

(A) $ab < \dfrac{c^2}{4}$ (B) $ab \geq \dfrac{c^2}{4}$
(C) $ab \geq \dfrac{c}{4}$ (D) None of these

5. If $0 < \alpha < \beta < \dfrac{\pi}{2}$ then

(A) $\dfrac{\tan\beta}{\tan\alpha} < \dfrac{\alpha}{\beta}$ (B) $\dfrac{\tan\beta}{\tan\alpha} > \dfrac{\alpha}{\beta}$
(C) $\dfrac{\tan\alpha}{\tan\beta} < \dfrac{\alpha}{\beta}$ (D) $\dfrac{\tan\alpha}{\tan\beta} > \dfrac{\alpha}{\beta}$

6. If $a < 0$, the function $(e^{ax} + e^{-ax})$ is a monotonic decreasing function for all values of x, where

(A) $x > 0$ (B) $x < 0$
(C) $x > 1$ (D) $x < 1$

7. The range of values of a for which the function
$f(x) = x^3 + (a + 2)x^2 + 3ax + 5$
may be monotonic in R, is

(A) $a < 1$ (B) $1 < a < 4$
(C) $a > 4$ (D) None of these

8. The values of k for which the function
$f(x) = kx^3 - 9x^2 + 9x + 3$ may be increasing on R are

(A) $k > 3$ (B) $k < 3$
(C) $k \leq 3$ (D) None of these

9. The least possible value of k for which the function $f(x) = x^2 + kx + 1$ may be increasing on [1, 2] is

(A) 2 (B) – 2
(C) 0 (D) None of these

10. If $f(x) = 2x^3 + 9x^2 + \lambda x + 20$ is a decreasing function of x in the largest possible interval $(-2, -1)$ then λ is equal to

(A) 12 (B) – 12
(C) 6 (D) None of these

11. Let $f'(x) > 0$ and $g'(x) < 0$ for all $x \in R$. Then,

(A) $f[g(x)] > f[g(x-1)]$
(B) $f[g(x)] > f[g(x+1)]$
(C) $g[f(x)] > g[f(x-1)]$
(D) $g[f(x)] < g[f(x+1)]$

12. If the function $f(x) = 3\cos|x| - 6ax + b$ increases for all $x \in R$, then the range of values of a is given by

(A) $a > -\frac{1}{2}$ (B) $a < -\frac{1}{2}$
(C) $a \le b$ (D) $a \ge b$

13. The equation $x + e^x = 0$ has

(A) only one real root
(B) only two real roots
(C) no real root
(D) None of these

14. The value of a in order that $f(x) = \sin x - \cos x - ax + b$ decreases for all real values is given by

(A) $a \ge \sqrt{2}$ (B) $a < \sqrt{2}$
(C) $a \ge 1$ (D) $a < 1$

15. Let f and g be increasing and decreasing functions respectively from $[0, \infty)$ to $[0, \infty)$. Let $h(x) = f[g(x)]$. If $h(0) = 0$, then $h(x)$ is

(A) always zero (B) always negative
(C) always positive (D) strictly increasing

16. If $f''(x) < 0\ \forall\ x \in (a, b)$, then $f'(x) = 0$

(A) exactly once in (a, b)
(B) atmost once in (a, b)
(C) atleast once in (a, b)
(D) None of these

17. The two tangents to the curve $ax^2 + 2hxy + by^2 = 1$, $a > 0$ at the points where it crosses x-axis, are

(A) parallel
(B) perpendicular
(C) inclined at an angle $\frac{\pi}{4}$
(D) None of these

18. The curve $y - e^{xy} + x = 0$ has a vertical tangent at the point

(A) (1, 1) (B) at no point
(C) (0, 1) (D) (1, 0)

19. The set of all values of a for which the function $f(x) = (a^2 - 3a + 2)(\cos^2 x/4 - \sin^2 x/4) + (a-1)x + \sin 1$ does not possess critical points is

(A) $[1, \infty)$ (B) $(0, 1) \cup (1, 4)$
(C) $(-2, 4)$ (D) $(1, 3) \cup (3, 5)$

20. Let $f(x) = \begin{cases} -x^3 + \log_2 b & 0 < x < 1 \\ 3x & x \ge 1 \end{cases}$. Then set of values of b for which $f(x)$ has least value at $x = 1$ is:

(A) R^+ (B) (0, 16]
(C) $[16, \infty)$ (D) None of these

21. If at any point on a curve the sub-tangent and sub-normal are equal, then the length of the normal is equal to

(A) $\sqrt{2}$ ordinate (B) ordinate
(C) $\sqrt{2 \text{ ordinate}}$ (D) None of these

22. Tangent is drawn to the ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$, where $\theta \in (0, \theta/2)$. Then, the value of θ such that sum of intercepts on axes made by this tangent is minimum, is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{6}$
(C) $\frac{\pi}{8}$ (D) $\frac{\pi}{4}$

23. The minimum value of $a\tan^2 x + b\cot^2 x$ equals the maximum value of $a\sin^2\theta + b\cos^2\theta$ where $a > b > 0$, when

(A) $a = b$ (B) $a = 2b$
(C) $a = 3b$ (D) $a = 4b$

24. A function f is such that $f'(a) = f''(a) = f'''(a) = \ldots = f^{(2n)}(a) = 0$ and f has a local maximum value b at $x = a$, if $f(x)$ is

(A) $(x-a)^{2n+2}$
(B) $b - 1 - (x + 1 - a)^{2n-1}$
(C) $b - (x-a)^{2n+2}$
(D) $(x-a)^{2n+2} - b$

25. If $P = x^3 - \frac{1}{x^3}$ and $Q = x - \frac{1}{x}$, $x \in (0, x)$ then minimum value of P/Q^2

(A) is $2\sqrt{3}$ (B) is $-2\sqrt{3}$
(C) does not exist (D) None of these

26. If the area of the triangle included between the axes and any tangent to the curve $x^n y = a^n$ is constant, then n is equal to

(A) 1 (B) 2

(C) $\frac{3}{2}$ (D) $\frac{1}{2}$

27. If $f(x)$ and $g(x)$ are differentiable functions for $0 \le x \le 1$ such that $f(0) = 2$, $g(0) = 0$, $f(1) = 6$, $g(1) = 2$, then in the interval (0, 1),

(A) $f'(x) = 0$ for all x
(B) $f'(x) = 2g'(x)$ for atleast one x
(C) $f'(x) = 2g'(x)$ for atmost one x
(D) None of these

28. If $y = \frac{\sin(x+a)}{\sin(x+b)}$; $a \ne b$, then y has

(A) maximum at $x = 0$
(B) minimum at $x = 0$
(C) neither maximum nor minimum
(D) None of these

29. For a differentiable curve $y = f(x)$ having atleast two extremum in the interval $[a, b]$,

(A) two of its maximum values occur successively
(B) two of its minimum values occur successively
(C) maximum and minimum values occur alternatively
(D) None of the above

30. The points on the curve $xy^2 = 1$ which are nearest to the origin are

(A) $\left[\left(\frac{1}{2}\right)^{1/3}, \pm\left(\frac{1}{2}\right)^{-1/6}\right]$ (B) $\left[\left(\frac{1}{2}\right)^{1/3}, 2^{-1/6}\right]$

(C) $\left(2^{1/3}, \pm\left(\frac{1}{2}\right)^{-1/6}\right)$ (D) None of these

31. N characters of information are held on magnetic tape, in batches of x characters each; the batch processing time is $\alpha + \beta x^2$ seconds; α, β are constants. The optimum value of x for fast processing is

(A) $\frac{\alpha}{\beta}$ (B) $\frac{\beta}{\alpha}$

(C) $\sqrt{\frac{\alpha}{\beta}}$ (D) $\sqrt{\frac{\beta}{\alpha}}$

32. AB is a diameter of a circle and C is any point on the circumference of the circle, then

(A) area of ΔABC is maximum when it is an isosceles
(B) area of ΔABC is minimum when it is an isosceles
(C) the perimeter of ΔABC is minimum when it is isosceles
(D) the perimeter of ΔABC is maximum when it is isosceles

33. Let $f(x) = 1 + 3x^2 + 3^2x^4 + ... + 3^{30} \cdot x^{60}$. Then $f(x)$ has

(A) atleast one maximum
(B) exactly one maximum
(C) atleast one minimum
(D) exactly one minimum

34. A function f is such that $f'(4) = f''(4) = 0$ and f has minimum value 10 at $x = 4$. Then $f(x) =$

(A) $4 + (x-4)^4$ (B) $10 + (x-4)^4$
(C) $(x-4)^4$ (D) None of these

35. The range of values of k for which the function $f(x) = (k^2 - 7k + 12)\cos x + 2(k-4)x + \log 2$ does not possess critical points, is

(A) (1, 5) (B) $(1, 5) - \{4\}$
(C) (1, 4) (D) None of these

36. The minimum value of the function

$f(x) = \frac{x^p}{p} + \frac{x^{-q}}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$ is

(A) 1 (B) 0
(C) 2 (D) None of these

37. If $f(x) = \frac{x^2-1}{x^2+1}$, for every real number x, then the minimum value of f

(A) does not exist because f is unboundecd
(B) is not attained even though f is bounded
(C) is equal to 1
(D) is equal to – 1

38. If a differentiable function $f(x)$ has a relative minimum at $x = 0$, then the function $y = f(x) + ax + b$ has a relative minimum at $x = 0$ for

(A) all $a > 0$ (B) all $b > 0$
(C) all a and b (D) all b if $a = 0$

39. On the curve $x^3 = 12y$, the abscissa changes at a faster rate than the ordinate. Then, x belongs to the interval

(A) (– 4, 4) (B) (–3, 3)
(C) (–2, 2) (D) None of these

40. The maximum value of radius vector where $\frac{c^4}{r^2} = \frac{a^2}{\sin^2 t} + \frac{b^2}{\cos^2 t}$; $(a, b > 0)$ is

(A) $(a+b)^2$ (B) $\frac{c^4}{(a+b)^2}$

(C) $\frac{c^2}{a+b}$ (D) $c^2(a+b)$

41. Let $f(x)$ and $g(x)$ be defined and differentiable for $x \geq x_0$ and $f(x_0) = g(x_0)$, $f'(x) > g'(x)$ for $x > x_0$, then

(A) $f(x) < g(x), x > x_0$ (b) $f(x) = g(x), x > x_0$
(C) $f(x) > g(x), x > x_0$ (d) None of these

42. If α and β ($\alpha < \beta$) be two different real roots of the equation $ax^2 + bx + c = 0$, then

(A) $\alpha > -\frac{b}{2a}$ (B) $\beta < -\frac{b}{2a}$
(C) $\alpha < -\frac{b}{2a} < \beta$ (D) $\beta < -\frac{b}{2a} < \alpha$

43. If $f'(x) = \frac{1}{1+x^2}$ for all x and $f(0) = 0$, then

(A) $f(2) < 0.4$ (B) $f(2) > 2$
(C) $0.4 < f(2) < 2$ (D) $f(2) = 2$

44. The interval in which λ should be if $f(x) = \sin^3 x + \lambda \sin^2 x(-\pi/2 < x < \pi/2)$ has exactly one maximum and one minimum is

(A) $(-1, 1)$ (B) $\left(-\frac{1}{2}, \frac{1}{2}\right)$
(C) $\left(\frac{-3}{2}, \frac{3}{2}\right)$ (D) $\left(\frac{-3}{2}, 0\right) \cup \left(0, \frac{3}{2}\right)$

45. Twenty metre of wire is available to fence off a flower bed in the form of a sector. If the flower bed has the maximum surface then radius is

(A) 10 (B) 5/2
(C) 5 (D) 15/2

46. If $f''(x) > 0$, $\forall x \in R$, $f'(3) = 0$ and $g(x) = f(\tan^2 x - 2\tan x + 4)$, $0 < x < \pi/2$, then $g(x)$ is increasing in

(A) $\left(0, \frac{\pi}{4}\right)$ (B) $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
(B) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (D) $\left(0, \frac{\pi}{2}\right)$

47. The normal to the curve $x = a(1 + \cos\theta)$, $y = a\sin\theta$ at θ always passes through the fixed point

(A) (a, a) (B) $(a, 0)$
(C) $(0, a)$ (D) None of these

48. If the tangent to the curve $2y^3 = ax^2 + x^3$ at the point (a, a) cuts off intercepts α and β on the coordinate axes such that $\alpha^2 + \beta^2 = 61$, then $a =$

(A) ± 30 (B) ± 5
(C) ± 6 (D) ± 61

49. If $x \in [0, 2]$ and $g(x) = f(x) + f(2 - x)$. Also, $f''(x) < 0$, then $g(x)$

(A) increases in [0, 2] (B) decreases in [0, 2]
(C) decreases in [0, 1) and increases in (1, 2]
(D) increases in [0, 1) and decreases in (1, 2]

50. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 40 minutes after the leakage began is

(A) 9/7 (B) 7/9
(C) 2/9 (D) 9/2

51. Let $a, b \in R$ be such that the fucntion f given by $f(x) = \ln|x| + bx^2 + ax$, $x \neq 0$ has extreme values at $x = -1$ and $x = 2$.

Statement 1: f has local maximum at $x = -1$ and at $x = 2$.

Statement 2: $a = \frac{1}{2}$ and $b = \frac{-1}{4}$.

(A) Statement-1 is false, Statement-2 is true.
(B) Statement-1 is true, statement-2 is true, statement-2 is a correct explanation for Statement-1.
(C) Statement-1 is true, statement-2 is true; statement-2 is **not** a correct explanation for Statement-1.
(D) Statement-1 is true, statement-2 is false.

52. Each side of a square is increasing at the uniform rate of 1 m/sec. If after some time the area of the square is increasing at the rate of 8 m^2/sec, then the area of square at that time in sq. meters is:

(A) 4 (B) 9
(C) 16 (D) 25

53. Let $a, b, c \in R$, $a > 0$ and function $f: R \to R$ be defined by $f(x) = ax^2 + bx + c$

Statement 1: $b^2 < 4ac \Rightarrow f(x) > 0$, for every value of x.

Statement 2: f is strictly decreasing in the interval $\left(-\infty, \frac{-b}{2a}\right)$ and strictly increasing in the interval $\left(\frac{-b}{2a}, \infty\right)$.

(A) Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is true, Statement-2 is true, Statement-2 is **not** a correct explanation for Statement-1.
(C) Statement-1 is true, Statement-2 is false.
(D) Statement-1 is false, Statement-2 is true.

54. How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have?

(A) 7 (B) 1
(C) 3 (D) 5

55. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval [–1, 1]

(A) $P(-1)$ is the minimum and $P(1)$ is the maximum of P
(B) $P(-1)$ is not minimum but $P(1)$ is the maximum of P
(C) $P(-1)$ is the minimum and $P(1)$ is not the maximum of P
(D) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

56. Let f: [2, 7] $\to$ [0, ∞) be a continuous and differentiable function. Then,

$$(f(7)-f(2))\,\frac{(f(7)^2+(f(2))^2+f(2)f(7)}{3}$$ is equal to

(A) $5f^2(c)f'(C)$ (B) $5f'(c)$
(C) $f(c)f'(C)$ (D) None of these
where $c \in (2, 7)$.

57. The fraction exceeding its pth power by the greatest number possible, where $p \geq 2$, is

(A) $\left(\frac{1}{p}\right)^{1/p-1}$ (B) $\left(\frac{1}{p}\right)^{p-1}$
(C) $p^{1/p-1}$ (D) None of these

58. Let the function $f(x)$ be defined as

$$f(x) = \begin{cases} \tan^{-1}\alpha - 3x^2, & 0 < x < 1 \\ -6x, & x \geq 1 \end{cases}$$

$f(x)$ can have a maximum at $x = 1$ if the value of α is
(A) 0 (B) 2
(C) 1 (D) None of these

59. Let $f(x) = \begin{cases} |x|, & 0 < |x| \leq 2 \\ 1, & x = 0 \end{cases}$. Then, at $x = 0$, f has

(A) a local maximum (B) no local maximum
(C) a local minimum (D) no extremum

60. Let $f(x) = \int_0^x \frac{\cos t}{t}\,dt$ $(x > 0)$; then for $x = (2n + 1)\frac{\pi}{2}$, $f(x)$ has

(A) minima when $n = 0, 2, 4, ...$
(B) maxima when $n = 0, 2, 4, 6, ...$
(C) neither max. nor min. when $n = -1, -3, -5, ...$
(D) None of these

61. If the equation $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x = 0$ has a positive root $x = \alpha$, then the equation
$na_n x^{n-1} + (n-1)\,a_{n-1}x^{n-2} + ... + a_1 = 0$ has a positive root, which is

(A) smaller than α
(B) greater than α
(C) equal to α
(D) greater than or equal to α

62. If a, b, c be non-zero real numbers such that

$$\int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx$$
$$= \int_0^2 (1+\cos^8 x)(ax^2+bx+c)\,dx = 0,$$

then the equation $ax^2 + bx + c = 0$ will have

(A) one root between 0 and 1 and other root between 1 and 2
(B) both the roots between 0 and 1
(C) both the roots between 1 and 2
(D) None of these

63. Let f be a function which is continuous and differentiable for all real x. If $f(2) = -4$ and $f'(x) \geq 6$ for all $x \in [2, 4]$, then

(A) $f(4) < 8$ (B) $f(4) \geq 8$
(C) $f(4) \geq 12$ (D) None of these

64. If $ax + \frac{b}{x} \geq c$ for all positive x, where $a, b > 0$, then

(A) $ab < \frac{c^2}{4}$ (B) $ab \geq \frac{c^2}{4}$
(C) $ab \geq \frac{c}{4}$ (D) None of these

65. Let f be *a* continuous, diferentiable and bijective function. If the tangent to $y = f(x)$ at $x = a$ is also the normal to $y = f(x)$ at $x = b$, then there exists at least one $c \in (a, b)$ such that

(A) $f'(c) = 0$ (B) $f'(c) > 0$
(C) $f'(c) < 0$ (D) None of these

66. The values of k for which the function
$f(x) = kx^3 - 9x^2 + 9x + 3$ may be increasing on R are
(A) $k > 3$ (B) $k < 3$
(C) $k \leq 3$ (D) None of these

67. The least possible value of k for which the function $f(x) = x^2 + kx + 1$ may be increasing on [1, 2] is
(A) 2 (B) – 2
(C) 0 (D) None of these

68. Let $a + b = 4$, $a < 2$ and $g(x)$ be a monotonically increasing function of x.

Then, $f(a) = \int_0^a g(x)\,dx + \int_0^b g(x)\,dx$

(A) increases with increase in $(b - a)$
(B) decreases with increase in $(b - a)$

(C) increases with decrease in $(b - a)$
(D) None of these

69. The equation $x + e^x = 0$ has
(A) only one real root
(B) only two real roots
(C) no real root
(D) None of these

70. The value of a in order that
$f(x) = \sin x - \cos x - ax + b$
decreases for all real values is given by
(A) $a \geq \sqrt{2}$ (B) $a < \sqrt{2}$
(C) $a \geq 1$ (D) $a < 1$

71. If $f''(x) < 0 \ \forall \ x \in (a, b)$, then $f'(x) = 0$
(A) exactly once in (a, b)
(B) at most once in (a, b)
(C) at least once in (a, b)
(D) None of these

72. The minimum value of $a \tan^2 x + b \cot^2 x$ equals the maximum value of $a \sin^2 \theta + b \cos^2 \theta$ where $a > b > 0$, when
(A) $a = b$ (B) $a = 2b$
(C) $a = 3b$ (D) $a = 4b$

73. If $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$, for every real number, then minimum value of f
(A) Does not exist
(B) Is note attained even through f is bounded
(C) Is equal to 1
(D) Is equal to –1

74. If $y = a \log|x| + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then
(A) $a = 2, b = -1$
(B) $a = 2, b = -1/2$
(C) $a = -2, b = 1/2$
(D) None of these

75. If $f(x)$ and $g(x)$ are differentiable functions for $0 \leq x \leq 1$ such that $f(0) = 2$, $g(0) = 0$, $f(1) = 6$, $g(1) = 2$, then in the interval $(0, 1)$,
(A) $f'(x) = 0$ for all x
(B) $f'(x) = 2g'(x)$ for at least one x
(C) $f'(x) = 2g'(x)$ for at most one x
(D) None of these

76. The difference between the greatest and least values of the function $f(x) = \cos x + \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x$ is
(A) 2/3 (B) 8/7
(C) 9/4 (D) 3/8

77. A function f is such that $f'(4) = f''(4) = 0$ and f has minimum value 10 at $x = 4$. Then $f(x) =$
(A) $4 + (x - 4)^4$ (B) $10 + (x - 4)^4$
(C) $(x - 4)^4$ (D) None of these

78. The range of values of k for which the function
$f(x) = (k^2 - 7k + 12) \cos x + 2(k - 4)x + \log 2$
does not possess critical points, is
(A) $(1, 5)$ (B) $(1, 5) - \{4\}$
(C) $(1, 4)$ (D) None of these

79. If a differentiable function $f(x)$ has a relative minimum at $x = 0$, then the function $y = f(x) + ax + b$ has a relative minimum at $x = 0$ for
(A) all $a > 0$ (B) all $b > 0$
(C) all a and b (D) all b if $a = 0$

80. Let $f(x)$ and $g(x)$ be defined and differentiable for $x \geq x_0$ and $f(x_0) = g(x_0)$, $f'(x) > g'(x)$ for $x > x_0$, then
(A) $f(x) < g(x), x > x_0$ (b) $f(x) = g(x), x > x_0$
(C) $f(x) > g(x), x > x_0$ (d) None of these

81. If α and β $(\alpha < \beta)$ be two different real roots of the equation $ax^2 + bx + c = 0$, then
(A) $\alpha > -\dfrac{b}{2a}$ (B) $\beta < -\dfrac{b}{2a}$
(C) $\alpha < -\dfrac{b}{2a} < \beta$ (D) $\beta < -\dfrac{b}{2a} < \alpha$

82. If $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $|p(x)| \leq |e^{x-1} - 1|$ for all $x \geq 0$, then $|a_1 + 2a_2 + 3a_3 + \ldots + na_n|$
(A) ≤ 1 (B) ≥ 1
(C) ≥ 0 (D) ≤ 0

83. The maximum value of radius vector where
$\dfrac{c^4}{r^2} = \dfrac{a^2}{\sin^2 t} + \dfrac{b^2}{\cos^2 t}$; $(a, b > 0)$ is
(A) $(a + b)^2$ (B) $\dfrac{c^4}{(a + b)^2}$
(C) $\dfrac{c^2}{a + b}$ (D) $c^2(a + b)$

84. Let $f(x) = \begin{cases} -x^3 + \log_2 b & 0 < x < 1 \\ 3x & x \geq 1 \end{cases}$. Then, the set of values of b for which $f(x)$ has least value at $x = 1$ is
(A) R^+ (B) $(0, 16]$
(C) $[16, \infty)$ (D) None of these

85. The second drivative $f''(x)$ of the function $f(x)$ exists for all x in $[0, 1]$ and satisfies $|f''(x)| \leq 1$. If $f(0) = f(1)$, then for all x in $[0, 1]$
(A) $|f'(x)| < 1$ (B) $|f'(x)| > 1$
(C) $|f'(x)| = 1$ (D) $f(x)$ is constant

86. Let the function f be defined as

$$f(x) = \begin{cases} \dfrac{P(x)}{x-2}, & x \neq 2 \\ 7, & x = 2 \end{cases}$$

where $P(x)$ is a polynomial such that $P'''(x)$ is identically equal to 0 and $P(3) = 9$. If $f(x)$ is continuous at $x = 2$, then

(A) $P(x) = 2x^2 - x - 6$ (B) $P(x) = 2x^2 + x - 6$
(C) $P(x) = 2x^2 - x + 6$ (D) None of these

87. The equation $x^5 - 3x - 1 = 0$ has, in the interval [1, 2]

(A) at least one root (B) at most one root
(C) no root (D) a unique root

88. If the equation $x - \sin x = k$ has a unique root in $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, then the range of values of k are

(A) $\left(1-\dfrac{\pi}{2}, \dfrac{\pi}{2}-1\right)$ (B) $\left[1-\dfrac{\pi}{2}, \dfrac{\pi}{2}-1\right]$
(C) $\left[0, \dfrac{\pi}{2}+1\right]$ (D) None of these

89. The largest term in the sequence

$a_n = \dfrac{n}{n^2+10}$, $n \in N$ is

(A) $\dfrac{4}{26}$ (B) $\dfrac{3}{19}$
(C) $\dfrac{7}{18}$ (D) None of these

90. The range of values of a for which all roots of the equation $3x^4 + 4x^3 - 12x^2 + a = 0$ are real and distinct is

(A) (0, 5) (B) (1, 4)
(C) (–1, 5) (D) None of these

91. If $\phi(x) = f(x) + f(1-x)$ and $f''(x) < 0$ in (–1, 1), then $\phi(x)$ strictly increases in the interval

(A) $\left(0, \dfrac{1}{2}\right)$ (B) $\left(\dfrac{1}{2}, 1\right)$
(C) (–1, 0) (D) (0, 1)

92. $f(x)$ is a cubic function with $f(1) = -6$, $f(-1) = 10$ and has maxima at $x = -1$. If $f'(x)$ has minima at $x = 1$, then

(A) $f(x) = x^3 + 3x^2 - 9x + 5$
(B) $f(x) = x^3 - 3x^2 - 9x + 5$
(C) $f(x) = x^3 - 3x^2 + 9x + 5$
(D) $f(x) = x^3 - 3x^2 - 9x + 5$

93. If the function

$$f(x) = \begin{cases} -x^3 + \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}, & 0 \le x < 1 \\ 2x - 3, & 1 \le x \le 3 \end{cases}$$

has the least value at $x = 1$, then all possible real values of b are

(A) (–1, 1) (B) $(-2, -1) \cup [1, \infty)$
(C) (–2, 1) (D) None of these

94. The function $f(x) = \dfrac{|x+1|}{x^2}$ is strictly decreasing in the interval

(A) $(-\infty, -2) \cup (0, 1)$ (B) $(-2, 0) \cup (1, \infty)$
(C) $(-2, -1) \cup (0, \infty)$ (D) None of these

95. If the equation $ax^2 + bx + c = 0$, $a, b, c, \in R$ has at least one root in (0, 1), then

(A) $2a + 3b + 6c = 0$ (B) $a + 3b + 6c = 0$
(C) $2a + b + 6c = 0$ (D) $2a + 3b + c = 0$

96. The range of values of a so that the equation $x^3 - 3x + a = 0$ has three real and distinct roots is

(A) $(-\infty, -2) \cup (2, \infty)$ (B) (– 2, 0)
(C) (– 2, 0) (D) (– 2, 2)

97. The curves $\dfrac{x^2}{a} + \dfrac{y^2}{b} = 1$ and $\dfrac{x^2}{a_1} + \dfrac{y^2}{b_1} = 1$ will cut orthogonally if

(A) $a + b = a_1 + b_1$ (B) $a - b = a_1 - b_1$
(C) $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{a_1} - \dfrac{1}{b_1}$ (D) None of these

98. If x and y are the sides of two squares such that $y = x - x^2$, then the rate of change of the area of the second square with respect to the first square is

(A) $2x^2 - 3x + 1$ (B) $2x^2 + 3x + 1$
(C) $2x^2 - 3x - 1$ (D) $2x^2 + 3x - 1$

99. The point on the curve $3x^2 - 4y^2 = 72$ which is nearest to the line $3x + 2y + 1 = 0$ is

(A) (6, – 3) (B) (6, 3)
(C) (– 6, 3) (D) (– 6, – 3)

100. If the function $f(x) = (a^2 - 3a + 2)\cos\dfrac{x}{2} + (a-1)x$ possesses critical points, then a belongs to the interval

(A) $(-\infty, 0) \cup (4, \infty)$
(B) $(-\infty, 0] \cup [4, \infty)$
(C) $(-\infty, 0] \cup \{1\} \cup [4, \infty)$
(D) None of these

101. If the function $f(x) = \int_0^x |\log_2(\log_3(\log_4(\cos t + a)))|\,dt$, be increasing for all real values of x, then

(A) $a \geq 2$ (B) $a \geq 5$
(C) $a < 5$ (D) $a < 2$

102. The value of n, for which the function $f(x) = (x^2 - 4)^n (x^2 - x + 1)$, $n \in N$ assumes a local minima at $x = 2$, is

(A) an even number
(B) an odd number
(C) an irrational number
(D) cannot be determined

103. If the function $f(x) = \left(1 - \frac{\sqrt{21 - 4b - b^2}}{b + 1}\right)x^3 + 5x + \sqrt{16}$ increases for all x, then

(A) $b \in (-1, 2)$
(B) $b \in (-7, 3) - \{-1\}$
(C) $b \in (-7, -1) \cup (2, 3)$
(D) None of these

104. The range of parameter b, for which the function

$$f(x) = \int_0^x (bt^2 + b + \cos t)\,dt$$

is entirely increasing or decreasing for all real values of x is

(A) $[-1, 1]$ (B) $(-\infty, -1] \cup [1, \infty)$
(C) $(-\infty, -1) \cup (1, \infty)$ (D) $(-1, 1)$

105. Let $f(x) = (x - 3)(x - 4)(x - 4)(x - 5)(x - 6)$, then

(A) $f'(x)$ has four roots
(B) three roots of $f'(x) = 0$ lie in $(3, 4) \cup (4, 5) \cup (5, 6)$
(C) the equation $f'(x) = 0$ has only one root
(D) three roots of $f'(x) = 0$ lie in $(2, 3) \cup (3, 4) \cup (4, 5)$

106. For $a \in [\pi, 2\pi]$ and $n \in Z$, the critical points of $f(x) = \frac{1}{3}\sin a \tan^3 x + (\sin a - 1)\tan x + \sqrt{\frac{a - 2}{8 - a}}$ are

(A) $x = n\pi$ (B) $x = 2n\pi$
(C) $x = (2n + 1)\pi$ (D) None of these

107. Let $f''(x) > 0 \ \forall\, x \in R$ and $g(x) = f(2 - x) + f(4 + x)$. Then, $g(x)$ is increasing in

(A) $(-\infty, -1)$ (B) $(-\infty, 0)$
(C) $(-1, \infty)$ (D) None of these

108. The curves $x^2 - 4y^2 + c = 0$ and $y^2 = 4x$ will cut orthogonally for

(A) $c \in (0, 16)$ (B) $c \in (-3, 4)$
(C) $c \in (3, 4)$ (D) None of these

109. Which of the following is not true?

The function $f(x) = x^2 + \frac{\lambda}{x}$ has a

(A) minimum at $x = 2$ if $\lambda = 16$
(B) maximum at $x = 2$ if $\lambda = 16$
(C) maximum for no real value of λ
(D) point of inflexion at $x = 1$ if $\lambda = -1$

110. If the parabola $y = f(x)$, having axis parallel to the y-axis, touches the line $y = x$ at $(1, 1)$, then

(A) $2f'(0) + f(0) = 1$ (B) $2f(0) + f'(0) = 1$
(C) $2f(0) - f'(0) = 1$ (D) $2f'(0) - f(0) = 1$

111. The angle between the tangents at any point P and the line joining P to the origin O, where P is a point on the curve $\ln(x^2 + y^2) = c\tan^{-1}\frac{y}{x}$, c is a constant

(A) varies as $\tan^{-1}x$ (B) varies as $\tan^{-1}y$
(C) is a constant (D) None of these

112. If the equation $ax^2 + bx + c = 0$ has two distinct positive roots, then the equation $ax^2 + (b + 6a)x + (c + 3b) = 0$ has

(A) two positive roots
(B) exactly one positive root
(C) at least one positive root
(D) no positive root

113. If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) then there exists at least one $c \in (a, b)$ such that $\frac{f(b) - f(a)}{b^3 - a^3}$ equals

(A) $3c^2 f'(C)$ (B) $\frac{f'(c)}{3c^2}$
(C) $f(c)f'(C)$ (D) None of these

114. Let $f(x) = \ln x$ and $g(x) = x^2$. If $c \in (4, 5)$, then $c\,ln\left(\frac{4^{25}}{5^{16}}\right)$ equals

(A) $c\ln 5 - 8$ (B) $2(c^2\ln 4 - 8)$
(C) $2(c^2\ln 5 - 8)$ (D) $c\ln 4 - 8$

More than One Option Correct Type

115. If $0 < x < \frac{\pi}{2}$, then

(A) $\frac{2}{\pi} > \frac{\sin x}{x}$ (B) $\frac{2}{\pi} < \frac{\sin x}{x}$

(C) $\frac{\sin x}{x} < 1$ (D) $\frac{\sin x}{x} > 1$

116. If $0 < x < \frac{\pi}{2}$, then

(A) $\cos(\sin x) > \cos x$
(B) $\cos(\sin x) < \cos x$
(C) $\cos(\sin x) > \sin(\cos x)$
(D) $\cos(\sin x) < \sin(\cos x)$

117. $(1 + x)^p \le 1 + x^p$, where

(A) $p > 1$ (B) $0 \le p \le 1$
(C) $x > 0$ (D) $x < 0$

118. The function $f(x) = |x + 2| + |x - 1|$ is

(A) increasing in $(1, \infty)$
(B) increasing in $[1, \infty)$
(C) decreasing in $(-\infty, -2]$
(D) decreasing in $(-\infty, -2)$

119. If $g(x) = f(x) + f(1 - x)$ and $f''(x) < 0$ for $0 \le x \le 1$, then

(A) $g(x)$ increases in $\left(-\infty, \frac{1}{2}\right)$

(B) $g(x)$ increases in $\left(0, \frac{1}{2}\right)$

(C) $g(x)$ decreases in $\left(\frac{1}{2}, 1\right)$

(D) $g(x)$ decreases in $\left(\frac{1}{2}, \infty\right)$

120. The function $f(x) = \frac{|x-1|}{x^2}$

(A) increases in $(-\infty, 0) \cup (1, 2)$
(B) increases in $(0, 1) \cup (2, \infty)$
(C) decreases in $(0, 1) \cup (2, \infty)$
(D) decreases in $(-\infty, \infty) \cup (1, 2)$

121. Let $h(x) = f(x) - [f(x)]^2 + [f(x)]^3$ for every real number x. Then

(A) h is increasing whenever f is increasing
(B) h is increasing whenever f is decreasing
(C) h is decreasing whenever f is decreasing
(D) nothing can be said in general

122. Given that $f'(x) > g'(x)$ for all real x and $f(0) = g(0)$, then

(A) $f(x) > g(x)\ \forall\ x \in (0, \infty)$
(B) $f(x) < g(x)\ \forall\ x \in (-\infty, 0)$
(C) $f(x) < g(x)\ \forall\ x \in (0, \infty)$
(D) $f(x) > g(x)\ \forall\ x \in (-\infty, 0)$

123. For the function $f(x) = \int_2^x e^{-t^4/4}(4 - t^2)\, dt$,

(A) maximum occurs at $x = 2$
(B) minimum occurs at $x = -2$
(C) maximum occurs at $x = -2$
(D) minimum occurs at $x = 2$

124. Let $f(x) = \sin x + \frac{1}{2}\cos 2x$. Then

(A) $\min_{x \in \left[0, \frac{\pi}{2}\right]} f(x) < \frac{4}{3}$

(B) $\min_{x \in \left[0, \frac{\pi}{2}\right]} f(x) > \frac{3}{4}$

(C) $\min_{x \in \left[0, \frac{\pi}{2}\right]} f(x) > \frac{2}{3}$

(D) $\min_{x \in \left[0, \frac{\pi}{2}\right]} f(x) < \frac{3}{2}$

125. If $(x - a)^{2n}(x - b)^{2m+1}$, where m and n are positive integers and $a > b$, is the derivative of a function f, then

(A) $x = a$ gives neither a maximum nor a minimum
(B) $x = a$ gives a maximum
(C) $x = b$ gives a minimum
(D) $x = b$ gives neither a maximum nor a minimum

126. If $f(x) = |x| + |x - 1| + |x - 2|$, then

(A) $f(x)$ has minima at $x = 1$
(B) $f(x)$ has maxima at $x = 0$
(C) $f(x)$ has neither maxima nor minima at $x = 0$
(D) $f(x)$ has neither maxima nor minima at $x = 2$

127. $(1 + x)^p \le 1 + x^p$, where

(A) $p > 1$ (B) $0 \le p \le 1$
(C) $x > 0$ (D) $x < 0$

128. If $g(x) = f(x) + f(1 - x)$ and $f''(x) < 0$ for $0 \le x \le 1$, then

(A) $g(x)$ increases in $\left(-\infty, \frac{1}{2}\right)$

(B) $g(x)$ increases in $\left(0, \frac{1}{2}\right)$

(C) $g(x)$ decreases in $\left(\frac{1}{2}, 1\right)$

(D) $g(x)$ decreases in $\left(\frac{1}{2}, \infty\right)$

129. The function $f(x) = \frac{|x-1|}{x^2}$

(A) increases in $(-\infty, 0) \cup (1, 2)$
(B) increases in $(0, 1) \cup (2, \infty)$
(C) decreases in $(0, 1) \cup (2, \infty)$
(D) decreases in $(-\infty, \infty) \cup (1, 2)$

130. Let $h(x) = f(x) - [f(x)]^2 + [f(x)]^3$ for every real number x. Then,

(A) h is increasing whenever f is increasing
(B) h is increasing whenever f is decreasing
(C) h is decreasing whenever f is decreasing
(D) nothing can be said in general

131. Given that $f'(x) > g'(x)$ for all real x and $f(0) = g(0)$, then

(A) $f(x) > g(x)\ \forall\ x \in (0, \infty)$
(B) $f(x) < g(x)\ \forall\ x \in (-\infty, 0)$
(C) $f(x) < g(x)\ \forall\ x \in (0, \infty)$
(D) $f(x) > g(x)\ \forall\ x \in (-\infty, 0)$

132. If $f'(x) > 0$ and $g'(x) < 0\ \forall\ x \in R$, then

(A) $fog(x) > fog(x+1)$
(B) $fog(x) > fog(x-1)$
(C) $gof(x) > gof(x+1)$
(D) $gof(x) > gof(x-1)$

133. The points on the curve $ay^2 = x^3$ where the normal line makes equal intercepts on the axes are

(A) $\left(\frac{2a}{9}, \frac{8a}{27}\right)$ (B) $\left(\frac{4a}{9}, \frac{8a}{27}\right)$

(C) $\left(\frac{4a}{9}, \frac{-8a}{27}\right)$ (D) $\left(\frac{4a}{9}, \frac{4a}{27}\right)$

134. The equation of the straight line which is tangent at one point and normal at another point to the curve $y = 8t^3 - 1$, $x = 4t^2 + 3$, is

(A) $\sqrt{2}x - y = \frac{89\sqrt{2}}{27} - 1$

(B) $\sqrt{2}x - y = \frac{89\sqrt{2}}{27} + 1$

(C) $\sqrt{2}x + y = \frac{89\sqrt{2}}{27} - 1$

(D) $\sqrt{2}x + y = \frac{89\sqrt{2}}{27} + 1$

135. Let $f(x) = \begin{cases} x+2, & -1 \le x < 0 \\ 1, & x = 0 \\ \frac{x}{2}, & 0 < x \le 1 \end{cases}$

Then, on $[-1, 1]$, $f(x)$ has
(A) a minimum
(B) a maximum
(C) neither a maximum nor a minimum
(D) $f'(0)$ does not exist

Passage Based Questions

Passage 1

Let $y = f_1(x)$ and $y = f_2(x)$ be the two curves, meeting at some point $P(x_1, y_1)$, then

θ = angle between the two curves at $P(x_1, y_1)$
= angle between the tangents to the curves at $P(x_1, y_1)$

Clearly, $\theta = \pm(\theta_1 - \theta_2)$,

where θ_1 and θ_2 are the inclinations of tangents to the curves $y = f_1(x)$ and $y = f_2(x)$ respectively at the point P.

Also, $$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2},$$

where, $m_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ for $y = f_1(x) = \tan\theta_1$

and, $m_2 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ for $y = f_2(x) = \tan\theta_2$

If the angle of intersection of two curves is a right angle, the two curves are said to be orthogonal.

If the two curves are orthogonal, then

$$\left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2} = -1.$$

136. The acute angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection when $x > 0$, is $\tan^{-1}(m)$, where $m =$

(A) $\frac{2\sqrt{2}}{7}$ (B) $\frac{4\sqrt{2}}{7}$

(C) $\frac{\sqrt{2}}{7}$ (D) None of these

Passage 2

If $f(x)$ be a function of x, where $f(x)$ is continuous in the closed interval $[a, b]$ and differentiable in the open interval (a, b). Also, $f(a) = f(b)$, i.e., the values at the end points a and$b$ are equal. Then, there exists at least one point c between a and b (i.e., $a < c < b$) for which $f'(c) = 0$. There may be more than one point in (a, b) at which $f'(x) = 0$. Geometrically, it means there exists at least one point c $(c, f(c))$ on the curve between the points, $A(a, f(a))$ and $B(b, f(b))$ at which the tangent is parallel to the x-axis.

137. If $a + b + c = 0$, then the equation $3ax^2 + 2bx + c = 0$ has, in the interval (0, 1)

(A) at least one root (B) at most one root
(C) no root (D) None of these

138. The equation $x \log x = 3 - x$ has, in the interval (1, 3)

(A) exactly one root (B) at least one root
(C) at most one root (D) no root

139. Between any two real roots of the equation $e^x \sin x = 1$, the equation $e^x \cos x = -1$ has

(A) at least one root
(B) exactly one root
(C) at most one root
(D) no root

Passage 3

If two functions f and g defined on $[a, b]$ are

1. continuous on the closed interval $[a, b]$
2. derivable on the open interval (a, b)
3. $g'(x) \neq 0$ for any $x \in (a, b)$

then there exists at least one real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

We may write it as

$$\frac{f(b) - f(a)}{g(b) - g(a)} g'(c) = f'(c)$$

Hence, there is an ordinate $x = c$ between $x = a$ and $x = b$ such that the tangents at the points, where $x = c$ cuts the graphs of the functions $f(x)$ and $\frac{f(b) - f(a)}{g(b) - g(a)} g(x)$, are mutually parallel.

140. The value of c for the functions $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ in the interval $[a, b]$ is

(A) $\sqrt{a}$ (B) $\sqrt{b}$
(C) $\sqrt{ab}$ (D) None of these

141. $\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = F(\alpha)$, where $0 < \alpha < \theta < p < \frac{\pi}{2}$.

Then, $F(\theta) =$

(A) $\tan\theta$ (B) $\cot\theta$
(C) $\sin\theta$ (D) $\cos\theta$

142. The value of c for the functions $f(x) = e^x$ and $g(x) = e^{-x}$ in the interval (a, b) is

(A) $\frac{a+b}{2}$ (B) $\frac{a+b}{4}$
(C) $a + b$ (D) None of these

Match the Column Type

143.

Column-I	Column-II
I. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and $m(b)$ the minimum value of $f(x)$ for a given b. As b varies, the range of $m(b)$ is	(A) (0, 1]
II. The set of values of x for which $\log(1 + x) < x$, is	(B) (0, 1)
III. If $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots + \frac{a_{n-1}}{2} + a_n = 0$, then the equation $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$ has at least one root in the interval	(C) $(0, \infty)$
IV. If $27a + 9b + 3c + d = 0$, then the equation $4ax^3 + 3bx^2 + 2cx + d = 0$ has at least one real root lying in the interval	(D) (0, 3)

144.

Column-I	Column-II
I. If $f(x) = \begin{cases} 4x - x^3 + \ln(a^2 - 3a + 3), 0 \le x < 3 \\ x - 18, \quad x \ge 3 \end{cases}$ = has a local minima at $x = 3$, than a belongs to	(A) $R - [-\sqrt{3}, \sqrt{3}]$
II. If the function $f(x) = \left(\frac{\sqrt{a+1}}{a-1} - 1\right)x^3 - x + \ln(a - 1)$ is strictly decreasing $\forall x \in R$, then a belongs to	(B) $(0, \infty)$
III. If the function $f(x) = x^3 + ax^2 + a^2x + 2\sin^2 x$ is strictly increasing $\forall x \in R$, then a belongs to	(C) $[1, 2]$
IV. The function $f(x) = \lvert e^{ax} - e^{-ax}\rvert$, $a > 0$ is strictly increasing in the interval	(D) $(3, \infty)$

Assertion-Reason Type

Instructions: In the following questions an Assertion (A) is given, followed by a Reason (R). Mark your responses from the following options:

(A) Assertion(A) is True and Reason(R) is True; Reason(R) is a correct explanation for Assertion(A)
(B) Assertion(A) is True, Reason(R) is True; Reason(R) is not a correct explanation for Assertion(A)
(C) Assertion(A) is True, Reason(R) is False
(D) Assertion(A) is False, Reason(R) is True

145. Assertion: If a quadratic curve touches the line $y = x$ at the point (1, 1), then the values of x for which the curve has a negative gradient are $x < \frac{1}{2}$

Reason: The equation of the curve is $y = x^2 - x + 1$

146. Assertion: The function $f(x) = \frac{\sin x}{x}$ is decreasing in the interval $\left(0, \frac{\pi}{2}\right)$

Reason: $\tan x > x$ for $0 < x < \frac{\pi}{2}$

147. Assertion: If $0 < x < \frac{\pi}{2}$, then $\frac{2}{\pi} < \frac{\sin x}{x} < 1$

Reason: $\tan x < x$ for $0 < x < \frac{\pi}{2}$

148. Assertion: If $0 < x < \frac{\pi}{2}$, then $\cos(\sin x) > \cos x > \sin(\cos x)$

Reason: $\sin x < x$ for $0 < x < \frac{\pi}{2}$.

149. Assertion: If $0 < \alpha < \beta < , \frac{\pi}{2}$ then $\frac{\tan\beta}{\tan\alpha} > \frac{\alpha}{\beta}$

Reason: $x\tan x$ is increasing for $0 < x < \frac{\pi}{2}$

150. Assertion: Let f and g be increasing and decreasing functions respectively from $[0, \infty]$ to $[0, \infty]$. Let $h(x) = f(g(x))$. If $h(0) = 0$, then $h(x)$ is always zero

Reason: $h(x)$ is an increasing function of x

151. Assertion: If $f'(x) = \frac{1}{1+x^2}$ for all x and $f(0) = 0$, then $0.4 < f(2) < 2$

Reason: By mean value theorem, there exists a point $c \in (0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

152. Assertion: If $f(x) = \tan x$, $x \in \left[0, \frac{\pi}{7}\right]$, then $\frac{\pi}{7} < f\left(\frac{\pi}{7}\right) < \frac{2\pi}{7}$

Reason: $\sec^2 x$ is strictly increasing in $\left[0, \frac{\pi}{7}\right]$

153. Assertion: For $b > a > 1$, $\frac{1}{b\ln b} < \frac{f(b) - f(a)}{b - a} < \frac{1}{a\ln a}$, where $f(x) = \ln(\ln x)$, $x > 1$

Reason: $\frac{1}{x\ln x}$ is strictly decreasing in (a, b)

154. **Assertion:** $\sin(\tan x) \geq x,\ \forall\ x \in \left[0, \frac{\pi}{4}\right]$

Reason: $1 - \cos x \leq \frac{x^2}{2}$

155. **Assertion:** $(a + b)^{1/n} \leq a^{1/n} + b^{1/n}$, where $a, b \geq 0$ and $n \geq 1$

Reason: The function $f(x) = (1 + x)^p - x^p - 1, x \geq 0$ and $0 < p \leq 1$ decreases in $(0, \infty)$

156. **Assertion:** $303^{202} < 202^{303}$

Reason: The function $f(x) = \frac{\ln x}{x}$ strictly increases in (e, ∞).

157. **Assertion:** $\ln(\cos\theta) < \cos(\ln\theta)$,

where $e^{-\theta/2} < \theta < \frac{\pi}{2}$

Reason: $\ln x < x\ \forall\ x > 0$

Previous Year's Questions

158. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$: **[2002]**

(A) cut at right angle (B) touch each other

(C) cut at an angle $\frac{\pi}{3}$ (D) cut at an angle $\frac{\pi}{4}$

159. The function $f(x) = \cot^{-1} x + x$ increases in the interval: **[2002]**

(A) $(1, \infty)$ (B) $(-1, \infty)$

(C) $(-\infty, \infty)$ (D) $(0, \infty)$

160. The greatest value of $f(x) = (x + 1)^{1/3} - (x - 1)^{1/3}$ on $[0, 1]$ is: **[2002]**

(A) 1 (B) 2 (C) 3 (D) 1/3

161. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals **[2003]**

(A) 3 (B) 1 (C) 2 (D) $\frac{1}{2}$

162. A function $y = f(x)$ has a second order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point (2, 1) and at that point the tangent to the graph is $y = 3x - 5$, then the function is **[2004]**

(A) $(x - 1)^2$ (B) $(x - 1)^3$

(C) $(x + 1)^3$ (D) $(x + 1)^2$

163. The normal to the curve $x = a(1 + \cos\theta)$, $y = a\sin\theta$ at θ always passes through the fixed point **[2004]**

(A) $(a, 0)$ (B) $(0, a)$ (C) $(0, 0)$ (D) (a, a)

164. The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point θ is such that **[2005]**

(A) It passes through the origin

(B) It makes angle $\frac{\pi}{2} + \theta$ with the x-axis

(C) It passes through $\left(a\frac{\pi}{2}, -a\right)$

(D) It is at a constant distance from the origin

165. A function is matched below against an interval where it is supposed to be increasing. Which of the following pairs is incorrectly matched? **[2005]**

Interval	**Function**
(A) $(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$
(B) $[2, \infty)$	$2x^3\ 3x^2 - 12x + 6$
(C) $\left(-\infty, \frac{1}{3}\right]$	$3x^2 - 2x + 1$
(D) $(-\infty, -4]$	$x^3 + 6x^2 + 6$

166. Let f be differentiable for all x. If $f(1) = -2$ and $f'(x) \geq 2$ for $x \in [1, 6]$, then **[2005]**

(A) $f(6) \geq 8$ (B) $f(6) < 8$

(C) $f(6) < 5$ (D) $f(6) = 5$

167. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness than melts at a rate of 50 cm^3/min. When the thickness of ice is 5 cm, then the rate at which the thickness of ice decreases, is **[2005]**

(A) $\frac{1}{36\pi}$ cm/min (B) $\frac{1}{18\pi}$ cm/min

(C) $\frac{1}{54\pi}$ cm/min (D) $\frac{5}{6\pi}$ cm/min

168. If the equation $a_nx^n + a_{n-1}x^{n-1} + - + a_1x = 0$, $a_1 \neq 0$, $n \geq 2$, has a positive root $x = \alpha$, then the equation $na_nx^{n-1} + (n - 1)a_{n-1}x^{n-2} + \ldots + a_1 = 0$ has a positive root, which is **[2005]**

(A) greater than a

(B) smaller than a

(C) greater than or equal to a

(D) equal to a

169. The function $f(x) = \frac{x}{2} + \frac{2}{x}$ has a local minimum at **[2006]**

(A) $x = 2$ (B) $x = -2$

(C) $x = 0$ (D) $x = 1$

170. Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is **[2006]**

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{4}$

171. The normal to a curve at $P(x, y)$ meets the x-axis at G. If the distance of G from the origin is twice the abscissa of P, then the curve is a **[2007]**

(A) ellipse (B) parabola
(C) circle (D) hyperbola

172. A value of C for which the conclusion of Mean Value Theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3] is **[2007]**

(A) $2 \log_3 e$ (B) $\frac{1}{2}\log_e 3$
(C) $\log_3 e$ (D) $\log_e 3$

173. The equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this line from which the other tangent to the parabola is perpendicular to the given tangent is **[2007]**

(A) (–1, 1) (B) (0, 2)
(C) (2, 4) (D) (–2, 0)

174. Suppose the cube $x^3 - px + q$ has three distinct real roots where $p > 0$ and $q > 0$. Then which one of the following holds? **[2008]**

(A) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$

(B) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$

(C) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

(D) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

175. How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have? **[2008]**

(A) 7 (B) 1 (C) 3 (D) 5

176. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval [–1, 1] **[2009**

(A) $P(-1)$ is the minimum and $P(1)$ is the maximum of P
(B) $P(-1)$ is not minimum but $P(1)$ is the maximum of P
(C) $P(-1)$ is the minimum and $P(1)$ is not the maximum of P
(D) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

177. The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is **[2009]**

(A) $\frac{3\sqrt{2}}{8}$ (B) $\frac{2\sqrt{3}}{8}$
(C) $\frac{3\sqrt{2}}{5}$ (D) $\frac{\sqrt{3}}{4}$

178. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, which is parallel to the x-axis, is **[2010]**

(A) $y = 1$ (B) $y = 2$
(C) $y = 3$ (D) $y = 0$

179. Let $f: R \to R$ be defined by $f(x) = \begin{cases} k - 2x, & \text{if } x \le -1 \\ 2x + 3 & \text{if } x > -1 \end{cases}$.
If f has a local minimum at $x = -1$, then apossible value of k is **[2010]**

(A) 0 (B) $-\frac{1}{2}$ (C) –1 (D) 1

180. A spherical balloon is filled with 4500 π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72 π cubic meters per minute, then the rate (in meters per minute) at which thera-dius of the balloon decreases 49 minutes after the leakage began is **[2012]**

(A) $\frac{9}{7}$ (B) $\frac{7}{9}$ (C) $\frac{2}{9}$ (D) $\frac{9}{2}$

181. Let the real values a, b be such that the function f given by $f(x) = \ln |x| + bx^2 + ax$, $x \neq 0$ has extreme values at $x = -1$ and $x = 2$. **[2012]**

Statement 1: f has local maximum at $x = -1$ and at $x = 2$.

Statement 2: $a = \frac{1}{2}$ and $b = \frac{-1}{4}$

(A) Statement 1 is false, statement 2 is true
(B) Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1
(C) Statement 1 is true, statement 2 is true; statement 2 is not a correct explanation for statement 1
(D) Statement 1 is true, statement 2 is false

182. If the functions f and g are differentiable functions on [0, 1] satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in]0,1[$ **[2014]**

(A) $2f'(C) = g'(c)$ (B) $2f'(C) = 3g'(c)$
(C) $f'(C) = g'(c)$ (D) $f'(C) = 2g'(c)$

183. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$, then **[2014]**

(A) $\alpha=-6, \beta=\frac{1}{2}$ (B) $\alpha=-6, \beta=-\frac{1}{2}$

(C) $\alpha=2, \beta=-\frac{1}{2}$ (D) $\alpha=2, \beta=\frac{1}{2}$

184. The normal to the curve, $x^2+2xy-3y^2=0$, at (1, 1) : **[2015]**

(A) meets the curve again in the second quadrant.
(B) meets the curve again in the third quadrant.
(C) meets the curve again in the fourth quadrant.
(D) does not meet the curve again.

185. A wire of length 2 units is cur into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then: **[2016]**

(A) $2x = r$ (B) $2x = (\pi + 4)r$
(C) $(4 - \pi)x = \pi r$ (D) $x = 2r$

186. Consider $f(x)=\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$, $x \in \left(0, \frac{\pi}{2}\right)$.

A normal to $y=f(x)$ at $x=\frac{\pi}{6}$ also passes through the point: **[2016]**

(A) $\left(\frac{\pi}{4}, 0\right)$ (B) (0, 0)

(C) $\left(0, \frac{2\pi}{3}\right)$ (D) $\left(\frac{\pi}{6}, 0\right)$

ANSWER KEYS

Single Option Correct Type

1. (B)	**2.** (A)	**3.** (B)	**4.** (B)	**5.** (B)	**6.** (A)	**7.** (B)	**8.** (A)	**9.** (B)	**10.** (A)
11. (B)	**12.** (B)	**13.** (A)	**14.** (A)	**15.** (A)	**16.** (B)	**17.** (A)	**18.** (D)	**19.** (B)	**20.** (C)
21. (A)	**22.** (B)	**23.** (D)	**24.** (C)	**25.** (C)	**26.** (A)	**27.** (B)	**28.** (C)	**29.** (C)	**30.** (A)
31. (C)	**32.** (A)	**33.** (D)	**34.** (B)	**35.** (B)	**36.** (A)	**37.** (D)	**38.** (D)	**39.** (C)	**40.** (C)
41. (C)	**42.** (C)	**43.** (C)	**44.** (D)	**45.** (C)	**46.** (C)	**47.** (B)	**48.** (A)	**49.** (D)	**50.** (C)
51. (B)	**52.** (C)	**53.** (B)	**54.** (B)	**55.** (B)	**56.** (A)	**57.** (A)	**58.** (D)	**59.** (A)	**60.** (B)
61. (A)	**62.** (A)	**63.** (B)	**64.** (B)	**65.** (A)	**66.** (A)	**67.** (B)	**68.** (A)	**69.** (A)	**70.** (A)
71. (B)	**72.** (D)	**73.** (D)	**74.** (B)	**75.** (B)	**76.** (C)	**77.** (B)	**78.** (B)	**79.** (D)	**80.** (C)
81. (C)	**82.** (A)	**83.** (C)	**84.** (C)	**85.** (A)	**86.** (A)	**87.** (D)	**88.** (B)	**89.** (B)	**90.** (A)
91. (A)	**92.** (B)	**93.** (B)	**94.** (C)	**95.** (A)	**96.** (D)	**97.** (B)	**98.** (A)	**99.** (C)	**100.** (C)
101. (B)	**102.** (A)	**103.** (C)	**104.** (B)	**105.** (B)	**106.** (D)	**107.** (C)	**108.** (D)	**109.** (B)	**110.** (B)
111. (C)	**112.** (C)	**113.** (B)	**114.** (B)						

More than One Option Correct Type

115. (B) and (C)	**116.** (A) and (C)	**117.** (B) and (C)	**118.** (A) and (D)	**119.** (B) and (C)
120. (A) and (C)	**121.** (A) and (C)	**122.** (A) and (B)	**123.** (A) and (B)	**124.** (A) and (D)
125. (A) and (C)	**126.** (A), (C) and (D)	**127.** (B) and (C)	**128.** (B) and (C)	**129.** (A) and (C)
130. (A) and (C)	**131.** (A) and B)	**132.** (A) and C)	**133.** (B) and (C)	**134.** (B) and (C)
135. (C) and (D)				

Passage Based Questions

Passage 1

138. (B)

Passage 2

139. (B) **140.** (B) **141.** (A)

Passage 3

142. (C) **143.** (B) **144.** (A)

Match the Column Type

136. I $\rightarrow$ (A); II $\rightarrow$ (C); III $\rightarrow$ (B); IV $\rightarrow$ (D)
137. I $\rightarrow$ (C); II $\rightarrow$ (D); III $\rightarrow$ (A); IV $\rightarrow$ (B)

Assertion-Reason Type

145. (A) **146.** (A) **147.** (C) **148.** (A) **149.** (A) **150.** (C) **151.** (A) **152.** (A)
153. (A) **154.** (A) **155.** (A) **156.** (C) **157.** (A)

Previous Year's Questions

158. (A) **159.** (A) **160.** (C) **161.** (C) **162.** (B) **163.** (A) **164.** (D) **165.** (C) **166.** (A) **167.** (B)
168. (B) **169.** (A) **170.** (B) **171.** (A) and (D) **172.** (A) **173.** (D) **174.** (A) **175.** (B) **176.** (B)
177. (A) **178.** (C) **179.** (C) **180.** (C) **181.** (B) **182.** (D) **183.** (C) **184.** (C) **185.** (D) **186.** (C)

HINTS AND SOLUTIONS

Single Option Correct Type

1. Let $f(x) = \log(1 + x) - x$

$\Rightarrow\ f'(x) = \dfrac{1}{1+x} - 1 = -\dfrac{x}{1+x}$

$\Rightarrow\ f'(x) < 0$, for $x > 0$

$\Rightarrow\ f(x)$ is decreasing for $x > 0$

$\Rightarrow\ f(x) < f(0)$, for $x > 0$

$\Rightarrow\ \log(1 + x) - x < 0$, for $x > 0$

i.e., $\log(1 + x) < x$, for $x > 0$.

The correct option is (B)

2. Clearly, $f(x)$ is periodic with period 1 and f(x) has one local maximum in [0, 1]

The correct option is (A)

3. We have, $f(x) = \dfrac{\sin x}{x}$

$\Rightarrow\ f'(x) = \dfrac{x\cos x - \sin x}{x^2} = \dfrac{\cos x(x - \tan x)}{x^2}$

But $\tan x > x$ and $\cos x > 0$, for $0 < x < \dfrac{\pi}{2}$

$\therefore\ f'(x) < 0$ in the interval $.\left(0, \dfrac{\pi}{2}\right)$

Thus, $f(x)$ is decreasing in $\left(0, \dfrac{\pi}{2}\right)$.

The correct option is (B)

4. Let $f(x) = ax + \dfrac{b}{x} - c;\ x > 0;\ a, b > 0$

$\Rightarrow\ f'(x) = a - \dfrac{b}{x^2} = \dfrac{ax^2 - b}{x^2}$

$f'(x) = 0 \Rightarrow\ a - \dfrac{b}{x^2} = 0 \Rightarrow\ x = \left(\dfrac{b}{a}\right)^{1/2}$

But $ax + \dfrac{b}{x} \geq c;\ \therefore f(x) \geq 0$ for all $x > 0$

$\therefore\ f\left[\left(\dfrac{b}{a}\right)^{1/2}\right] \geq 0 \quad \Rightarrow\ a\left(\dfrac{b}{a}\right)^{1/2} + b\left(\dfrac{a}{b}\right)^{1/2} - c \geq 0$

$\Rightarrow\ 2\sqrt{ab} \geq c \Rightarrow\ ab \geq \dfrac{c^2}{4}$

The correct option is (B)

5. Let $f(x) = x \tan x$

$\Rightarrow\ f'(x) = \tan x \times 1 + x\sec^2 x > 0$, for $x \in \left(0, \dfrac{\pi}{2}\right)$.

So, $f(x)$ is increasing for $x \in \left(0, \dfrac{\pi}{2}\right)$.

Since, $0 < \alpha < \beta < .\dfrac{\pi}{2}\ \ \pi f(\alpha) > f(\beta)$

$\Rightarrow\ \alpha\tan\beta > \alpha\tan\beta$ i.e., $\dfrac{\tan\beta}{\tan\alpha} > \dfrac{\alpha}{\beta}$

The correct option is (B)

6. Let $f(x) = e^{ax} + e^{-ax}$

$\Rightarrow\ f'(x) = a(e^{ax} - e^{-ax}) = \dfrac{a(e^{2ax} - 1)}{e^{ax}}$

$f(x)$ is a decreasing function if $f'(x) < 0$

$\Rightarrow\ \dfrac{a(e^{2ax} - 1)}{e^{ax}} < 0 \quad \Rightarrow\ e^{2ax} - 1 < 0 \Rightarrow e^{2ax} < 1$

$\Rightarrow\ 2ax < 0 \quad \Rightarrow\ x > 0 \qquad (\because a < 0)$

Thus, $f(x)$ is a decreasing function for $x > 0$.

The correct option is (B)

7. We have, $f(x) = x^3 + (a + 2)x^2 + 3ax + 5$

$\Rightarrow\ f'(x) = 3x^2 + 2(a + x)x + 3a$

$f(x)$ is monotonic in R if $f'(x) < 0,\ \forall x \in R$

or $\ f'(x) > 0,\ \forall x \in R$

Since the coefficient of $x^2 = 3 > 0$, therefore, $f'(x) \nless 0\ \forall x \in R$.

$\therefore\ f'(x) > 0,\ \forall x \in R$

$\therefore\ $ Discriminant $= 4(a + 2)^2 - 4 \times 3 \times 3a < 0$

$\Rightarrow \quad a^2 - 5a + 4 < 0 \quad \Rightarrow \quad (a-1)(a-4) < 0$

i.e., $1 < a < 4$

The correct option is (B)

8. We have, $f(x) = kx^3 - 9x^2 + 9x + 3$

$\Rightarrow \quad f'(x) = 3kx^2 - 18x + 9$

Since $f(x)$ is increasing on R, therefore,

$f'(x) > 0, \forall x \in R$

$\Rightarrow \quad 3kx^2 - 18x + 9 > 0, \forall x \in R$

$\Rightarrow \quad kx^2 - 6x + 3 > 0, \forall x \in R$

$\Rightarrow \quad k > 0$ and $36 - 12k < 0$

$(\because ax^2 + bx + c > 0, \forall x \in R$

$\Rightarrow \quad a > 0$ and discriminant $< 0)$

$\Rightarrow \quad k > 3.$

Hence, $f(x)$ is increasing on R, if $k > 3$.

The correct option is (B)

9. We have, $f(x) = x^2 + kx + 1$

$\Rightarrow \quad f'(x) = 2x + k$. Also, $f''(x) = 2$.

Now, $f''(x) = 2, \forall x \in [1, 2]$

$\Rightarrow \quad f''(x) > 0, \forall x \in [1, 2]$

$\Rightarrow \quad f'(x)$ is an increasing function in the interval [1, 2]

$\Rightarrow \quad f'(1)$ is the least value of $f'(x)$ on [1, 2]

But $f'(x) > 0 \ \forall x \in [1, 2]$

$[\because f(x)$ is increasing on $(1, 2)]$

$\therefore \quad f'(1) > 0, \forall x \in [1, 2] \quad \Rightarrow \quad k > -2.$

Thus, the least value of k is -2.

The correct option is (B)

10. Since $f(x)$ is decreasing in the interval $(-2, -1)$, therefore,

$f'(x) < 0 \quad \Rightarrow \quad 6x^2 + 18x + \lambda < 0$

The value of λ should be such that the equation

$6x^2 + 18x + \lambda = 0$ has roots -2 and -1.

Therefore, $(-2)(-1) = \dfrac{\lambda}{6} \quad \Rightarrow \quad \lambda = 12$

The correct option is (A)

11. We have, $f'(x) > 0, \forall x \in$ R

$\Rightarrow \quad f(x)$ is increasing, $\forall x \in R$

$\therefore \quad f(x) < f(x+1)$ and $f(x) > f(x-1) \ \forall x \in R$

$\Rightarrow \quad g[f(x)] > g[f(x+1)]$ and $g[f(x)] < g[f(x-1)]$

as $\quad g(x)$ is decreasing, $\forall x \in R$

Also, $g'(x) < 0, \forall x \in R$

$\Rightarrow \quad g(x)$ is decreasing, $\forall x \in R$

$\Rightarrow \quad g(x) > g(x+1)$ and $g(x) < g(x-1), \forall x \in R$

$\therefore \quad f[g(x)] > f[g(x+1)]$ and $f[g(x)] < f[g(x-1), \forall x \in R$

as $\quad f(x)$ is increasing.

The correct option is (B)

12. We have,

$f(x) = 3\cos|x| - 6ax + b$

$= 3\cos x - 6ax + b \qquad (\because \cos(-x) = \cos x)$

$\Rightarrow \quad f'(x) = -3\sin x - 6a.$

Since $f(x)$ is an increasing function $\forall x \in R$

$\Rightarrow \quad f'(x) > 0 \ \forall x \in R$

$\Rightarrow \quad -3\sin x - 6a > 0 \ \forall x \in R$

In particular, at $x = \dfrac{\pi}{2}$

$-3 - 6a > 0 \quad \Rightarrow \quad a < -\dfrac{1}{2}$

The correct option is (B)

13. Let $f(x) = x + e^x = 0$.

Since $f(-\infty) = -\infty$ and $f(+\infty) = \infty$,

$\therefore \quad f(x) = 0$ has a real root.

Let the real root be α. Then $f(\alpha) = 0$.

Now, $f'(x) = 1 + e^x > 0, \forall x \in R$

$\therefore \quad f(x)$ is an increasing function $\forall x \in R$.

$\therefore \quad$ for any other real number α,

$f(\alpha) > f(\beta)$ or $f(\beta) < f(\alpha)$.

But $f(\alpha) = 0$; so, $f(\beta) \neq 0$.

$\therefore \quad f(x) = 0$ has no other real root.

Hence, the equation has only one real root.

The correct option is (A)

14. We have, $f(x) = \sin x - \cos x - ax + b$

$\Rightarrow \quad f'(x) = \cos x + \sin x - a$

$f(x)$ is a decreasing function for all real values of x, if $f'(x) < 0 \ \forall x \in R \quad \Rightarrow \quad \cos x + \sin x < a \ \forall x \in R$.

As the maximum value of $\cos x + \sin x$is $\sqrt{2}$, the above is possible when $a \geq \sqrt{2}$.

$\therefore \quad f(x)$ decreases for each $x \in R$ if $\boldsymbol{a} \geq \sqrt{2}$**.**

The correct option is (A)

15. Since$g(x)$ is decreasing,

$\therefore \quad g(x_2) \leq g(x_1)$ when $x_2 \geq x_1$.

Since $f(x)$ is increasing,

$\therefore \quad f[g(x_1)] \geq f[g(x_2)]$

$\Rightarrow \quad h(x_1) \geq h(x_2)$ when $x_2 \geq x_1$.

$\Rightarrow \quad h(x)$ is a decreasing function of x and $h(0) = 0$.

Also, domain of $h = [0, \infty)$ and range of $h = [0, \infty)$.

$\therefore \quad h(x) = 0, \forall x \in [0, \infty)$.

The correct option is (A)

16. Suppose, there are two points x_1 and $x_2$in (a, b) such that $f'(x_1) = f'(x_2) = 0$. By Rolle's theorem applied to f' on $[x_1, x_2]$, there must then be a $c \in (x_1, x_2)$ such that $f''(c) = 0$. This contradicts the given condition $f''(x) < 0 \ \forall x \in (a, b)$. Hence, our assumption is wrong. Therefore, there can be atmost one point in (a, b) at which $f'(x)$ is zero.

The correct option is (B)

17. When the curve meets x-axis, then

$y = 0 \quad \Rightarrow \quad ax^2 = 1 \quad \Rightarrow \quad x = \pm\dfrac{1}{\sqrt{a}}$

Hence, the points are $\left(\dfrac{1}{\sqrt{a}}, 0\right)$ and $\left(-\dfrac{1}{\sqrt{a}}, 0\right)$.

Differentiating the given equation with respect to x, we get

$$2ax + 2h\left(y + x\frac{dy}{dx}\right) + 2by\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{ax+hy}{hx+by}$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{\left(\frac{1}{\sqrt{a}},0\right)} = -\frac{a}{h} \text{ and } \left.\frac{dy}{dx}\right]_{\left(-\frac{1}{\sqrt{a}},0\right)} = -\frac{a}{h}$$

Hence, the tangents at these points are parallel.

The correct option is (A)

18. We have, $y - e^{xy} + x = 0$

$$\Rightarrow \quad 1 - e^{xy}\left(x + y\frac{dx}{dy}\right) + \frac{dx}{dy} = 0$$

Now, $\quad \dfrac{dx}{dy} = \dfrac{xe^{xy}-1}{1-ye^{xy}} = 0$ for vertical tangents

$\therefore \quad xe^{xy} - 1 = 0$

$\Rightarrow \quad e^{xy} = 1/x,\ y = 0 \quad \Rightarrow \quad x = 1$

The correct option is (D)

19. $f(x) = (a^2 - 3a + 2)(\cos^2 x/4 - \sin^2 x/4) + (a - 1)\,x + \sin 1$

$\Rightarrow \quad f(x) = (a - 1)\,(a - 2)\cos x/2 + (a - 1)x + \sin 1$

$$\Rightarrow \quad f'(x) = -\frac{1}{2}(a-1)(a-2)\sin\frac{x}{2} + (a-1)$$

$$\Rightarrow \quad f'(x) = (a-1)\left[1 - \frac{(a-2)}{2}\sin\frac{x}{2}\right]$$

If $f(x)$ does not possess critical points, then $f'(x) \neq 0$ for any $x \in \mathbb{R}$

$$\Rightarrow \quad (a-1)\left[1 - \frac{(a-2)}{2}\sin\frac{x}{2}\right] \neq 0 \text{ for any } x \in \mathbb{R}$$

$$\Rightarrow \quad a \neq 1 \text{ and } 1 - \left(\frac{a-2}{2}\right)\sin\frac{x}{2} = 0$$

must not have any solution in $\mathbb{R}$.

$$\Rightarrow \quad a \neq 1 \text{ and } \sin\frac{x}{2} = \frac{2}{a-2} \text{ is not solvable in } \mathbb{R}$$

$$\Rightarrow \quad a \neq 1 \text{ and } \left|\frac{2}{a-2}\right| > 1 \left[\begin{array}{l}\text{For } a = 2, f(x) = x + \sin 1\\ \therefore\ f'(x) = 1 \neq 0\end{array}\right]$$

$\Rightarrow \quad a \neq 1$ and $|a - 2| < 2$

$\Rightarrow \quad a \neq 1$ and $-2 < a - 2 < 2$

$\Rightarrow \quad a \neq 1$ and $0 < a < 4$

$\Rightarrow \quad a \in (0, 1) \cup (1, 4)$.

The correct option is (B)

20. Function f is decreasing before 1 and increasing after 1. f has least value at $x = 1$ if $\lim\limits_{x \to 1^-} f(x) \geq f(1)$

$\Rightarrow \quad -1 + \log_2 b \geq 3 \quad \Rightarrow \quad b \in [16, \infty)$

The correct option is (C)

21. We are given, sub-tangent = sub-normal

$$\Rightarrow \quad \left|\frac{y}{dy/dx}\right| = \left|y\frac{dy}{dx}\right| \Rightarrow \frac{dy}{dx} = \pm 1.$$

So, length of the normal = $\quad y\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

$= y\sqrt{2} \times = \sqrt{2}$ (ordinate).

The correct option is (A)

22. Equation of tangent is $\dfrac{x\sqrt{3}\cos\theta}{9} + y\sin\theta$

Now, sum of intercepts

$$(z) = a + b = \frac{9}{\sqrt{3}\cos\theta} + \frac{1}{\sin\theta}$$

$\Rightarrow \quad z = 3\sqrt{3}\sec\theta + \operatorname{cosec}\theta$

$$\Rightarrow \quad \frac{dz}{d\theta} = 3\sqrt{3}\sec\theta\tan\theta - \operatorname{cosec}\theta\cot\theta$$

Now, $3\sqrt{3}\sec\theta\tan\theta = \operatorname{cosec}\theta\cos\theta$

$$\Rightarrow \quad 3\sqrt{3}\frac{\sin\theta}{\cos^2\theta} = \frac{\cos\theta}{\sin^2\theta} \Rightarrow \tan^3\theta = \frac{1}{3\sqrt{3}}$$

$$\Rightarrow \quad \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$$

The correct option is (B)

23. Minimum value of $a\tan^2 x + b\cot^2 x$ is $2\sqrt{ab}$ and maximum value of $a\sin^2\theta + b\cos^2\theta$ is a

$[\therefore a\sin^2\theta + b\cos^2\theta = (a - b)\sin^2\theta + b]$

Given: $a = 2\sqrt{ab}$

$\therefore \quad a = 4b$

The correct option is (B)

24. Since $f^{2n+1}(a) = 0$, $f^{2n+2}(a) = -$ve and $f(a) = b$,

$\therefore \quad f(x) = -(x - a)^{2n+2} + b = b - (x - a)^{2n+2}$

The correct option is (C)

25. $P = x^3 - \dfrac{1}{x^3}$, $Q = x - \dfrac{1}{x}$

$$\therefore \quad \frac{P}{Q^2} = \frac{\left(x - \frac{1}{x}\right)\left(x^2 + 1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2}$$

$$= \frac{\left(x - \frac{1}{x}\right)^2 + 3}{\left(x - \frac{1}{x}\right)} = \left(x - \frac{1}{x}\right) + \frac{3}{\left(x - \frac{1}{x}\right)}$$

Clearly, the minimum does not exist.

The correct option is (C)

26. Let $P(x_1, y_1)$ be any point on the curve $x^n y = a^n$

Then, $\quad x_1^n y_1 = a^n \quad$ (1)

Now, the given curve is

$$\Rightarrow \quad nx^{n-1}y + x^n\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -n\frac{y}{x} \Rightarrow \left.\frac{dy}{dx}\right|_{(x_1,\,y_1)} = \frac{-ny_1}{x_1}$$

$$\Rightarrow \quad m_T = -n\frac{a^n}{x_1^{n+1}} \qquad \text{[using (1)]}$$

The equation of the tangent at $P(x_1, y_1)$ is

$$y - y_1 = \frac{na^n}{x_1^{n+1}}(x - x_1)$$

The tangent intersects the coordinate axes at

$$A\left(\frac{x_1^{n+1}y_1}{na^n} + x_1, 0\right) \text{ and } B\left(0, y_1 + \frac{na^n}{x_1^n}\right)$$

$\therefore$ Area of $\Delta AOB = \frac{1}{2}(OA \times OB)$

$$\Rightarrow \quad \Delta AOB = \frac{1}{2}\left(\frac{x_1^{n+1}y_1}{na^n} + x_1\right)\left(y_1 + \frac{na^n}{x_1^n}\right)$$

$$\Rightarrow \quad \Delta AOB = \frac{1}{2}\left(\frac{x_1}{n} + x_1\right)\left(\frac{a^n}{x_1^n} + \frac{na^n}{x_1^n}\right) \quad \text{[using (1)]}$$

$$\Rightarrow \quad \Delta AOB = \frac{1}{2}\frac{(n+1)^2}{n}a^n x_1^{1-n}$$

For the area to be a constant, we must have $1 - n = 0$ i.e., $n = 1$.

The correct option is (A)

27. Let $h(x) = f(x) - 2g(x), x \in [0, 1]$.

$\Rightarrow \quad h'(x) = f'(x) - 2g'(x)$

Also, $h(0) = f(0) - 2g(0) = 2 - 2 \times 0 = 2$.

$h(1) = f(1) - 2g(1) = 6 - 2 \times 2 = 2$.

$\therefore \quad h(0) = h(1)$.

Since $f(x)$ and $g(x)$ are differentiable in [0, 1], $h(x)$ is also differentiable in [0, 1]. Hence, $h(x)$ is also continuous in [0, 1].

So, all the conditions of Rolle's theorem are satisfied. Hence, there exists a point c, $0 < c < 1$ for which $h'(c) = 0$.

$\therefore \quad f'(c) - 2g'(c) = 0$ i.e., $f'(c) = 2g'(c)$.

The correct option is (B)

28. We have, $y = \dfrac{\sin(x+a)}{\sin(x+b)}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin(x+b)\cdot\cos(x+a)\cdot 1 - \sin(x+a)\cos(x+b)\cdot 1}{\sin^2(x+b)}$$

$$= \frac{\sin(b-a)}{\sin^2(x+b)} \neq 0 \text{ for any } x \text{ as } a \neq b.$$

Hence, y has neither maximum nor minimum.

The correct option is (C)

29. For a differentiable curve maximum or minimum occurs only at those points where the nature of the curve changes (i.e., if it is increasing then at that point it starts decreasing and vice-versa).

The correct option is (C)

30. Let $P(x, y)$ be such a point then $OP^2 = x^2 + y^2$.

Let $S = OP^2 = x^2 + \frac{1}{x} \Rightarrow \frac{dS}{dx} = 2x - \frac{1}{x^2}$

$$\therefore \quad \frac{dS}{dx} = 0 \Rightarrow \quad 2x - \frac{1}{x^2} = 0 \Rightarrow \quad x^3 = \frac{1}{2}$$

$$\therefore \quad x = \left(\frac{1}{2}\right)^{1/3}$$

Also, $\dfrac{d^2S}{dx^2} = 2 + \dfrac{2}{x^3} \Rightarrow \left.\dfrac{d^2S}{dx^2}\right]_{x=\left(\frac{1}{2}\right)^{1/3}} = 2 + \dfrac{2}{1/2} > 0$

$\therefore \quad S$ is minimum at $x = \left(\frac{1}{2}\right)^{1/3}$

Thus, for nearest point $x = \left(\frac{1}{2}\right)^{1/3}$ and then

$$y = \pm\left(\frac{1}{2}\right)^{-1/6}$$

The correct option is (A)

31. $T = \dfrac{N}{x}(\alpha + \beta x^2) = N\left(\dfrac{\alpha}{x} + \beta x\right)$

$$\Rightarrow \quad \frac{dT}{dx} = N\left(-\frac{\alpha}{x^2} + \beta\right)$$

$$\therefore \quad \frac{dT}{dx} = 0 \quad \Rightarrow \quad x^2 = \frac{\alpha}{\beta} \quad \Rightarrow \quad x = \sqrt{\frac{\alpha}{\beta}}$$

Also, $\dfrac{d^2T}{dx^2} = N\left(\dfrac{2\alpha}{x^3}\right) > 0$ at $x = \sqrt{\dfrac{\alpha}{\beta}}$

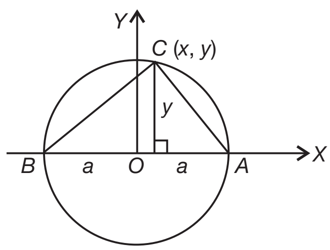
$\therefore \quad T$ is minimum for $x = \sqrt{\dfrac{\alpha}{\beta}}$

The correct option is (C)

32. Let the equation of the circle be $x^2 + y^2 = a^2$

Let $A(a, 0)$, $B(-a, 0)$ be the ends of the diameter and $C(x, y)$ be any point on the circle.

Then, area of ΔABC



$= A = \frac{1}{2} \times AB \times y$

$= ay = a\sqrt{a^2 - x^2}$

$\therefore$ A is maximum if $x = 0$

i.e., C lies on y-axis and then CAB is an isosceles triangle.

The correct option is (A)

33. We have,

$f(x) = 1 + 3x^2 + 3^2 \times x^4 + ... + 3^{30} \times x^{60}$

$\Rightarrow$ $f'(x) = x(6 + 4 \times 3^2 \times x^2 + ... + 60 \times 3^{30} \times x^{58})$

$f'(x) = 0 \Rightarrow x = 0.$

Also, $f''(x) = 1 \times (6 + 4 \times 3^2x^2 + ... + 60 \times 3^{30} \times x^{58})$
$+ x(4 \times 3^2 \times 2x + ... + 60 \times 3^{30} \times 58x^{57})$

$\Rightarrow$ $f''(0) = 6 > 0$. $\therefore f(x)$ has minimum at $x = 0$ only.

The correct option is (D)

34. Since $f'(4) = f''(4) = 0$, therefore,

$f(x) = (x - 4)^n + k$, where $n \geq 3$

But f has minimum at $x = 4$, so $n = 4$.

$\therefore$ $f(x) = (x - 4)^4 + k.$

Since $f(4) = 10$, therefore, $k = 10$.

Thus, $f(x) = (x - 4)^4 + 10$

The correct option is (B)

35. We have,

$f(x) = (k^2 - 7k + 12)\cos x + 2(k - 4)x + \log 2$

$= (k - 3)(k - 4)\cos x + 2(k - 4)x + \log 2$

$\Rightarrow$ $f'(x) = -(k - 3)(k - 4)\sin x + 2(k - 4)$

$= (k - 4)[-(k - 3)\sin x + 2]$

Since $f(x)$ does not have critical points, therefore $f'(x) = 0$ does not have any solution in R.

$\Rightarrow$ $k \neq 4$ and $2 - (k - 3)\sin x = 0$ is not solvable in R

$\Rightarrow$ $k \neq 4$ and $\sin x = \frac{2}{k-3}$ is not solvable in R

$\Rightarrow$ $k \neq 4$ and $\left|\frac{2}{k-3}\right| > 1$

$\Rightarrow$ $k \neq 4$ and $|k - 3| < 2$

$\Rightarrow$ $k \neq 4$ and $-2 < k - 3 < 2$

$\Rightarrow$ $k \neq 4$ and $1 < k < 5$ $\Rightarrow$ $k \in (1, 5) - \{4\}$.

The correct option is (B)

36. We have, $f(x) = \frac{x^p}{p} + \frac{x^{-q}}{q}$

$\Rightarrow$ $f'(x) = x^{p-1} - x^{-q-1}$

$= x^{p-1} - \frac{1}{x^{q+1}} = (x^{p+q} - 1)/x^{q+1}$

$f'(x) = 0$ gives $x^{p+q} = 1$ $\Rightarrow$ $x = 1$.

When $x < 1, f'(x) < 0$ and when $x > 1, f'(x) > 0$

Hence $f'(x)$ changes sign from negative to positive as x passes through the value $x = 1$ and hence $f(x)$ has a minimum value at $x = 1$.

The minimum value $= f(1) = \frac{1}{p} + \frac{1}{q} = \mathbf{1}$ (given).

The correct option is (A)

37. We have, $f(x) = \frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$.

$f(x)$ will be minimum when $\frac{2}{x^2+1}$ is maximum i.e. when $x^2 + 1$ is minimum i.e. at $x = 0$.

$\therefore$ Minimum value of $f(x) = f(0) = -1$.

The correct option is (D)

38. Since $f(x)$ has a relative minimum at $x = 0$

$\therefore$ $f'(0) = 0$ and $f''(0) > 0$.

Now, $y = f(x) + ax + b$

$\Rightarrow$ $\frac{dy}{dx} = f'(x) + a.$

$\therefore$ $\left.\frac{dy}{dx}\right]_{x=0} = f'(0) + a = a = 0$, if $a = 0$

Also, $\frac{d^2y}{dx^2} = f''(x)$. $\therefore$ $\left.\frac{d^2y}{dx^2}\right]_{x=0} = f''(0) > 0.$

$\therefore$ y has a relative minimum at $x = 0$, if $a = 0$ and for all b.

The correct option is (D)

39. By the given condition $\left|\frac{dy}{dy}\right| > 1$

Now, $x^3 = 12y$ $\Rightarrow$ $3x^2\frac{dx}{dy} = 12$

$\therefore$ $\frac{dx}{dy} = \frac{4}{x^2}$

$\therefore$ $\frac{4}{x^2} > 1$

$\Rightarrow$ $x^2 - 4 < 0$

$\Rightarrow$ $-2 < x < 2$

The correct option is (C)

40. $\frac{c^4}{r^2} = a^2\operatorname{cosec}^2 t + b^2\sec^2 t$

$\Rightarrow$ $r^2 = \frac{c^4}{a^2\operatorname{cosec}^2 t + b^2\sec^2 t}$

$\Rightarrow$ $r^2 = \frac{c^2}{a^2(1 + \cot^2 t) + b^2(1 + \tan^2 t)}$

$\Rightarrow$ $r^2 = \frac{c^4}{(a^2 + b^2) + (a^2\cot^2 t + b^2\tan^2 t)}$

$\Rightarrow$ $r^2 = \frac{c^4}{(a+b)^2 + (a\cot t - b\tan t)^2}$

Now r^2 is maximum if denominator is minimum,

$\therefore$ c is constant

The denominator is minimum if $(a \cot t - b \tan t)^2 = 0$

$\therefore$ Maximum value of $r^2 = \dfrac{c^4}{(a+b)^2}$

$\Rightarrow$ Maximum value of $r = \dfrac{c^2}{a+b}$

Hence, Maximum value of radius vector is $\left(\dfrac{c^2}{a+b}\right)$.

The correct option is (C)

41. Since $f(x_0) = g(x_0)$ and $f'(x) > g'(x)$, $\forall\, x > x_0$

$\therefore f'(x) - g'(x) > 0$

$\Rightarrow$ $f(x) - g(x)$ is an increasing function $\forall\, x > x_0$.

$\therefore$ $f(x) - g(x) > f(x_0) - g(x_0) = 0$

$\therefore$ $f(x) > g(x)$, $\forall\, x > x_0$.

The correct option is (C)

42. Let $f(x) = ax^2 + bx + c$.

Then $f(\alpha) = 0 = f(\beta)$.

Also, $f(x)$ is continuous and differentiable in $[\alpha, \beta]$ as it is a polynomial function of x.

Hence, by Rolle's theorem, there exists a k in (α, β), such that

$f'(k) = 0 \;\Rightarrow\; 2ak + b = 0 \;\Rightarrow\; k = -\dfrac{b}{2a}$

$\therefore$ $\alpha < -\dfrac{b}{2a} < \beta$

The correct option is (C)

43. We have, $f'(x) = \dfrac{1}{1+x^2}$ for all x. (1)

$\Rightarrow$ $f'(x) > 0$, for all x $(\because\ 1 + x^2 > 0)$

From (1), it follows that $f(x)$ is differentiable at all x, therefore, $f(x)$ is also continuous at all x.

$\therefore$ By mean value theorem in [0, 2]

$\dfrac{f(2) - f(0)}{2 - 0} = f'(c) = \dfrac{1}{1+c^2}$ where $0 < c < 2$

$\Rightarrow$ $\dfrac{f(2) - 0}{2} = \dfrac{1}{1+c^2}$ or $f(2) = \dfrac{2}{1+c^2}$ (2)

Now, $0 < c < 2$,

$\therefore$ $\dfrac{2}{1+c^2} < \dfrac{2}{1+0^2}$ or $\dfrac{2}{1+c^2} < 2$ (3)

and $\dfrac{2}{1+c^2} > \dfrac{2}{1+2^2} = \dfrac{2}{5} = 0.4$

or $\dfrac{2}{1+c^2} > 0.4$ (4)

From (2), (3) and (4) it follows that $0.4 < f(2) < 2$.

The correct option is (C)

44. $f(x) = \sin^3 x + \pi \sin^2 x$, $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

Now, $f'(x) = 3 \sin^2 x \cos x + 2\pi \sin x \cos x$

$\Rightarrow$ $f'(x) = 2 \sin x \cos x\left(\dfrac{3}{2}\sin x + \lambda\right)$

$\Rightarrow$ $f'(x) = \sin 2x\left(\dfrac{3}{2}\sin x + \lambda\right)$

For $f(x)$ to attain only one maxima or minima, we must have

$\dfrac{3}{2}\sin x + \lambda > 0\ \forall\, x \in$ $\Rightarrow$ $\lambda > -\dfrac{3}{2}\sin x$

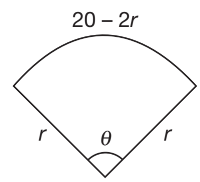
or $\lambda \in \left(-\dfrac{3}{2}, 0\right) \cup \left(0, \dfrac{3}{2}\right)$.

The correct option is (D)

45. Let the radius of the sector of the flower bed be r and the angle subtended be θ

$\therefore$ S = Surface Area of Sector

$\Rightarrow$ $S = \dfrac{\theta}{360}\pi r^2$



Also, $\theta = \dfrac{\text{Length of Area}}{\text{Radius}} = \dfrac{20 - 2r}{r}$

$\therefore$ $S = \dfrac{\theta}{2\pi}(\pi r^2) = \dfrac{20 - 2r}{2r}\cdot r^2 \Rightarrow S = (10 - r)r$

Now, $\dfrac{dS}{dr} = (10 - r) + r(-1)$

For surface area to be maximum

$\dfrac{dS}{dr} = 0 \Rightarrow r = 5$.

The correct option is (C)

46. $g'(x) = f'[(\tan x - 1)^2 + 3] \times 2(\tan x - 1)\sec^2 x$

Since $f''(x) > 0 \Rightarrow$ $f'(x)$ is increasing

So, $f'[(\tan x - 1)^2 + 3] > f'(3) = 0$, $\forall x \in \left(0, \dfrac{\pi}{4}\right) \cup \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

Also, $\tan x - 1 > 0$ $\forall x \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

So, $g(x)$ is increasing in $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$.

The correct option is (C)

47. The given curve is $x = a(1 + \cos\theta)$, $y = a\sin\theta$. The slope of the tangent at any point $P(x, y)$ i.e.,

$P[a(1 + \cos\theta), a\sin\theta]$ is $m_T = \left.\dfrac{dy}{dx}\right|_P$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$$

$\Rightarrow \quad m_T = -\cot\theta$, θ $m_N = \tan\theta$

The equation of the normal at

$[a(1 + \cos\theta), a\sin\theta]$ and slope m_N is

$y - a\sin\theta = \tan\theta\,[x - a(1 + \cos\theta)]$

$\Rightarrow \quad x\sin\theta - y\cos\theta = a\sin\theta$

Clearly, it passes through $(a, 0)$.

The correct option is (B)

48. The slope of the tangent at any point (a, a) on the curve is

$$m_T = \left.\frac{dy}{dx}\right|_{(a,\,a)}$$

$$\therefore \quad 6y^2\frac{dy}{dx} = 2ax + 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2ax + 3x^2}{6y^2}$$

$$\Rightarrow \quad m_T = \left.\frac{dy}{dx}\right|_{(a,\,a)} = \frac{5}{6}$$

The equation of the tangent at (a, a) is

$$y - a = \frac{5}{6}(x - a) \Rightarrow \quad 5x - 6y + a = 0$$

This cuts off intercepts of lengths $-\dfrac{a}{5}$ and $\dfrac{a}{6}$ with x and y-axis respectively. So, $\alpha = -\dfrac{a}{5}$ and $\beta = \dfrac{a}{6}$.

Since, $\alpha^2 + \beta^2 = 61$ (given)

$$\therefore \quad \frac{a^2}{25} + \frac{a^2}{36} = 61 \Rightarrow a^2 = 25 \times 36$$

$\therefore \quad a = \pm 30$

The correct option is (A)

49. Since, $g(x) = f(x) + f(2 - x)$, $0 \le x \le 2$ (1)

and $f''(x) < 0$, $0 \le x \le 2$ (2)

$\Rightarrow \quad g'(x) = f'(x) - f'(2 - x)$ (3)

$x - (2 - x) = 2(x - 1)$

From (2), $f'(x)$ is a decreasing function in $[0, 1)$, $x < 2 - x$

$\Rightarrow \quad f'(x) > f'(2 - x)$

$\therefore$ From (3), $g'(x) > 0$ and hence $g(x)$ increases in $[0, 1)$

In $(1, 2]$, $x > 2 - x \quad \Rightarrow \quad f'(x) < f'(2 - x)$

$\therefore \quad$ From (3), $g'(x) < 0$ and hence $g(x)$ decreases in $(1, 2]$.

The correct option is (D)

50. $V = \dfrac{4}{3}\pi r^3 \Rightarrow 4500\pi = \dfrac{4\pi r^3}{3}$

$$\frac{dV}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right) \Rightarrow 45 \times 25 \times 3 = r^3$$

$\Rightarrow \quad r = 15$ m

After 49 min = $(4500 - 49.72)\pi = 972\pi$ m^3

$$972\pi = \frac{4}{3}\pi r^3$$

$r^3 = 3 \times 243 = 3 \times 3^5$

$r = 9$

$$72\pi = 4\pi \times 9 \times 9\left(\frac{dr}{dt}\right)$$

$$\frac{dr}{dt} = \left(\frac{2}{9}\right)$$

The correct option is (C)

51. $f'(x) = \dfrac{1}{x} + 2bx + a$

at $x = -1$ $\quad -1 - 2b + a = 0$

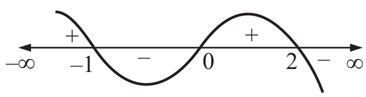
$a - 2b = 1$ (1)

at $x = 2$ $\quad \dfrac{1}{2} + 4b + a = 0$

$a + 4b = -\dfrac{1}{2}$ (2)

On solving Equation (1) and (2) $a = \dfrac{1}{2}$, $b = -\dfrac{1}{4}$

$$f'(x) = \frac{1}{x} - \frac{x}{2} + \frac{1}{2} = \frac{2 - x^2 + x}{2x} = \frac{-(x+1)(x-2)}{2x}$$



So maxima at $x = -1, 2$

The correct option is (B)

52. $\dfrac{dx}{dt} = 1$ m/sec

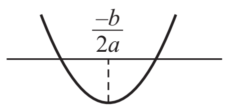
$\dfrac{d(A)}{dt} = 8$ m^2/sec

$$\Rightarrow \quad \frac{d}{dt}(x^2) = 8 \Rightarrow 2x\frac{dx}{dt} = 8 \Rightarrow \quad 2x = 8$$

$\Rightarrow \quad x = 4 \Rightarrow \quad A = 16$

The correct option is (C)

53. Statement-2 is true



$b^2 < 4ac \Rightarrow$ Disc. $< 0 \Rightarrow f(x) > 0$ for all $x \in R$

So Statement 1 is also true

The correct option is (B)

54. $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$

Let $f(x) = x^7 + 14x^5 + 16x^3 + 30x$

$\Rightarrow f'(x) = 7x^6 + 70x^4 + 48x^2 + 30 > 0\ \forall\ x.$

$\therefore$ $f(x)$ is an increasing function $\forall x$.

The correct option is (B)

55. $P(x) = x^4 + ax^3 + bx^2 + cx + d$

$P'(x) = 4x^3 + 3ax^2 + 2bx + c$

$\therefore x = 0$ is a solution for $P'(x) = 0, \Rightarrow c = 0$

$\therefore P(x) = x^4 + ax^3 + bx^2 + d$ (1)

Also, we have $P(-1) < P(1)$

$\therefore P'(x) = 0$, only when $x = 0$ and $P(x)$ is differentiable in $(-1, 1)$, we should have the maximum and minimum at the points $x = -1, 0$ and 1 only. Also, we have $P(-1) < P(1)$

$\therefore$ Max.of$P(x)$ = Max. $[P(0), P(1)]$

and Min. of $P(x)$ = Min. $[P(-1), P(0)]$

In the interval $[0, 1]$,

$P'(x) = 4x^3 + 3ax^2 + 2bx = x(4x^2 + 3ax + 2b)$

$\therefore P'(x)$ has only one root $x = 0$, $4x^2 + 3ax + 2b = 0$ has no real roots.

$\therefore (3a)^2 - 32b < 0 \Rightarrow \frac{3a^2}{32} < b$

$\therefore b > 0$

Thus, we have $a > 0$ and $b > 0$

$\therefore P'(x) = 4x^3 + 3ax^2 + 2bx > 0, \forall x \in (0, 1)$

Hence $P(x)$ is increasing in $[0, 1]$

$\therefore$ Max.of $P(x) = P(1)$

Similarly, $P(x)$ is decreasing in $[-1, 0]$

Therefore Min. $P(x)$ does not occur at $x = -1$

The correct option is (B)

56. Let $g(x) = f^3(x) \Rightarrow g'(x) = 3f^2(x) \cdot f'(x)$

$\therefore f : [2, 7] \to [0, \infty) \Rightarrow g : [2, 7] \to [0, \infty)$

Using LMV theorem on $g(x)$, we get

$g'(c) = \frac{g(7) - g(2)}{5}$; $c \in (2, 7)$

$\Rightarrow 5f^2(c)f'(c)$

$= (f(7) - f(2)).\frac{(f(7))^2 + (f(2))^2 + f(2)f(7)}{3}$

The correct option is (A)

57. Let $y = x - x^p$, where x is the fraction

$\Rightarrow \frac{dy}{dx} = 1 - px^{p-1}.$

For maximum or minimum, $\frac{dy}{dx} = 0$

$\Rightarrow 1 - px^{p-1} = 0 \Rightarrow x = \left(\frac{1}{p}\right)^{1/(p-1)}.$

Now, $\frac{d^2y}{dx^2} = -p(p-1)x^{p-2}$

$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=\left(\frac{1}{p}\right)^{1/(p-1)}} = -p(p-1)\left(\frac{1}{p}\right)^{p-2/(p-1)} < 0$

$\therefore$ y is maximum at $x = \left(\frac{1}{p}\right)^{1/(p-1)}.$

The correct option is (A)

58. We have,

$$f'(x) = \begin{cases} -6x, & 0 < x < 1 \\ -6, & x \geq 1 \end{cases}$$

$\therefore f'(1-h) = -6(1-h) < 0$

and, $f'(1+h) = -6 < 0.$

Since $f'(x)$ does not change sign as x passes through 1, therefore, $f(x)$ does not have a maximum or minimum at $x = 1$, whatever be the value of α.

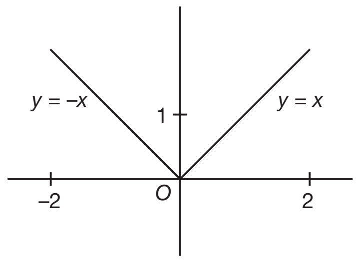
The correct option is (D)

59. We have,

$$f(x) = \begin{cases} -x, & -2 \leq x < 0 \\ 1, & x = 0 \\ x, & 0 < x \leq 2 \end{cases}$$

Clearly, from the graph,

$f(0) = 1, f(0-\varepsilon) < 1, f(0+\varepsilon) < 1$



where ε is small and positive.

$\therefore f(x)$ has a local maximum at $x = 0$.

The correct option is (A)

60. We have, $f'(x) = \frac{\cos x}{x}.$

$\therefore f'(x) = 0 \Rightarrow \cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}, n \in I.$

Also, $f''(x) = \frac{-x\sin x - \cos x}{x^2}.$

$$\therefore f''(x)\Big]_{x=(2n+1)\frac{\pi}{2}} = \frac{-(2n+1)\frac{\pi}{2}\sin(2n+1)\frac{\pi}{2} - 0}{\left[(2n+1)\frac{\pi}{2}\right]^2}$$

$$= \frac{-2(-1)^n}{(2n+1)\pi}.$$

< 0, for $n = 0, 2, 4, 6, ...$

$\therefore$ $f(x)$ has maxima when $n = 0, 2, 4, 6, ...$

The correct option is (B)

61. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x$.

Then, $f(\theta) = 0$ (Given). Also, $f(0) = 0$.

Moreover, $f(x)$ is continuous and differentiable in $[0, \theta]$ as it is a polynomial function of x. Hence, by Rolle's theorem, there exists a c in $(0, \theta)$ such that $f'(x) = 0$ for $x = c$, i.e.,

$na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + ... + 2a_2 x + a_1 = 0$

The correct option is (A)

62. Let $f(y) = \int_0^y (1+\cos^8 x)(ax^2+bx+c)\,dx$

$\Rightarrow$ $f'(y) = (1 + \cos^8 y)(ay^2 + by + c)$ (1)

Now, $f(1) = \int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx = 0$

and, $f(2) = \int_0^2 (1+\cos^8 x)(ax^2+bx+c)\,dx = 0$

Also, $f(0) = 0$

$\therefore$ $f(0) = f(1) = f(2)$.

Now, by Rolle's theorem for $f(x)$ in $[0, 1]$,

$f'(\theta) = 0$, for atleast one θ, $0 < \theta < 1$

and by Rolle's theorem for $f(x)$ in $[1, 2]$,

$f'(\theta) = 0$, for atleast one θ, $1 < \theta < 2$

From (1), $f'(\theta) = 0 \Rightarrow$ $(1 + \cos^8\theta)(a\theta^2 + b\theta + c) = 0$

But $1 + \cos^8\theta \neq 0$,

$\therefore$ $a\theta^2 + b\theta + c = 0$,

i.e., θ is a root of the equation $ax^2 + bx + c = 0$.

Similarly, $f'(\theta) = 0$ $\Rightarrow$ $a\theta^2 + b\theta + c = 0$,

i.e., θ is a root of the equation $ax^2 + bx + c = 0$.

But the equation $ax^2 + bx + c = 0$, being a quadratic equation, cannot have more than two roots.

$\therefore$ The equation $ax^2 + bx + c = 0$ has one root θ between 0 and 1 and other root θ between 1 and 2.

The correct option is (A)

63. By Mean Value theorem, there exists a real number $c \in (2, 4)$ such that

$$f'(c) = \frac{f(4)-f(2)}{4-2} \quad \Rightarrow \quad f'(c) = \frac{f(4)+4}{2}$$

Since $f'(x) \geq 6 \ \forall\, x \in [2, 4]$

$\therefore$ $f'(c) \geq 6$ $\Rightarrow$ $\dfrac{f(4)+4}{2} \geq 6$

$\Rightarrow$ $f(4) + 4 \geq 12$

$\Rightarrow$ $f(4) \geq 8$

The correct option is (B)

64. Let $f(x) = ax + \dfrac{b}{x} - c$; $x > 0$; $a, b > 0$

$\Rightarrow$ $f'(x) = a - \dfrac{b}{x^2} = \dfrac{ax^2 - b}{x^2}$

$f'(x) = 0$ $\Rightarrow$ $a - \dfrac{b}{x^2} = 0$ $\Rightarrow$ $x = \left(\dfrac{b}{a}\right)^{1/2}$

But $ax + \geq c$;

$\therefore$ $f(x) \geq 0$ for all $x > 0$

$\therefore$ $f\left[\left(\dfrac{b}{a}\right)^{1/2}\right] \geq 0$ $\Rightarrow$ $a\left(\dfrac{b}{a}\right)^{1/2} + b\left(\dfrac{a}{b}\right)^{1/2} - c \geq 0$

$\Rightarrow$ $2\sqrt{ab} \geq c$ $\Rightarrow$ $ab \geq \dfrac{c^2}{4}$

The correct option is (B)

65. Since the same line is tangent at one point $x = a$ and normal at other point $x = b$

$\Rightarrow$ Tangent at $x = b$ will be perpendicular to tangent at $x = a$

$\Rightarrow$ Slope of tangent changes from positive to negative or negative to positive. Therefore, it takes the value zero somewhere. Thus, there exists a point $c \in (a, b)$ where $f'(c) = 0$

The correct option is (A)

66. We have, $f(x) = kx^3 - 9x^2 + 9x + 3$

$\Rightarrow$ $f'(x) = 3kx^2 - 18x + 9$

Since $f(x)$ is increasing on R, therefore,

$f'(x) > 0 \ \forall x \in R$

$\Rightarrow$ $3kx^2 - 18x + 9 > 0 \ \forall\, x \in R$

$\Rightarrow$ $kx^2 - 6x + 3 > 0 \ \forall\, x \in R$

$\Rightarrow$ $k > 0$ and $36 - 12k < 0$

$[\because ax^2 + bx + c > 0 \ \forall\, x \in R \ \Rightarrow \ a > 0 \text{ and } D < 0]$

$\Rightarrow$ $k > 3$

Hence, $f(x)$ is increasing on R, if $k > 3$.

The correct option is (A)

67. We have, $f(x) = x^2 + kx + 1$

$\Rightarrow$ $f'(x) = 2x + k$. Also, $f''(x) = 2$

Now, $f''(x) = 2, \ \forall\, x \in [1, 2]$

$\Rightarrow$ $f''(x) > 0, \ \forall\, x \in [1, 2]$

$\Rightarrow$ $f'(x)$ is an increasing function in the interval $[1, 2]$.

$\Rightarrow$ $f'(1)$ is the least value of $f'(x)$ on $[1, 2]$

But $f'(x) > 0 \ \forall\, x \in [1, 2]$

$[f(x)$ is increasing on $[1, 2]]$

$\therefore$ $f'(1) > 0, \ \forall\, x \in [1, 2]$ $\Rightarrow$ $k > -2$

Thus, the least value of k is -2

The correct option is (B)

68. We have, $a + b = 4$

$\Rightarrow$ $b = 4 - a$ and $b - a = 4 - 2a = t$ (say)

Now, $f(a) = \int_0^a g(x)dx + \int_0^b g(x)dx = \int_0^a g(x)dx + \int_0^{4-a} g(x)dx$

$\Rightarrow$ $\dfrac{df(a)}{da} = g(a) - g(4-a)$

As $a < 2$ and $g(x)$ is increasing

$\Rightarrow \quad 4 - a > a \quad \Rightarrow \quad g(a) - g(4-a) < 0$

$\Rightarrow \quad \frac{df(a)}{da} < 0$.

Now, $\frac{df(a)}{da} = \frac{df(a)}{dt}.\frac{dt}{da} = -2.\ \frac{df(a)}{dt} \Rightarrow \frac{df(a)}{dt} > 0$.

Thus, $f(a)$ is an increasing funciton of t. Hence, the given expression increases with increase in $(b - a)$.

The correct option is (A)

69. Let $f(x) = x + e^x = 0$.

Since $f(-\infty) = -\infty$ and $f(+\infty) = \infty$,

$\therefore \quad f(x) = 0$ has a real root.

Let the real root be α. Then $f(\alpha) = 0$.

Now, $f'(x) = 1 + e^x > 0, \forall x \in R$

$\therefore \quad f(x)$ is an increasing function $\forall x \in R$.

$\therefore$ for any other real number β,

$f(\alpha) > f(\beta)$ or $f(\beta) < f(\alpha)$.

But $f(\beta) = 0$; so, $f(\beta) \neq 0$.

$\therefore \quad f(x) = 0$ has no other real root.

Hence, the equation has only one real root.

The correct option is (A)

70. We have, $f(x) = \sin x - \cos x - ax + b$

$\Rightarrow \quad f'(x) = \cos x + \sin x - a$

$f(x)$ is a decreasing function for all real values of x, if $f'(x) < 0\ \forall\ x \in R \quad \Rightarrow \quad \cos x + \sin x < a \forall x \in R$.

As the maximum value of $\cos x + \sin x$ is $\sqrt{2}$, the above is possible when $a \geq \sqrt{2}$.

$\therefore \quad f(x)$ decreases for each $x \in R$ if $\boldsymbol{a} \geq \sqrt{2}$

The correct option is (A)

71. Suppose, there are two points x_1 and x_2 in (a, b) such that $f'(x_1) = f'(x_2) = 0$. By Rolle's theorem applied to f' on $[x_1, x_2]$, there must then be a $c \in (x_1, x_2)$ such that $f''(c) = 0$. This contradicts the given condition $f''(x) < 0\ \forall\ x \in (a, b)$. Hence, our assumption is wrong. Therefore, there can be at most one point in (a, b) at which $f'(x)$ *is zero.*

The correct option is (B)

72. Minimum value of $a \tan^2 x + b \cot^2 x$ is $2\sqrt{ab}$ and maximum value of $a \sin^2 \theta + b \cos^2 \theta$ is a

$(\therefore \quad a \sin^2 \theta + b \cos^2 \theta = (a - b) \sin^2 \theta + b)$

Given: $a = 2\sqrt{ab}$.

$\therefore \quad a = 4b$

The correct option is (D)

73. We have $f(x) = 1 - \frac{2}{x^2+1}$

$f(x)$ will be minimum if $\frac{2}{x^2+1}$ is maximum, i.e., if $x^2 + 1$ is least i.e., when $x = 0$. Thus, minimum value of $f(x)$ is $f(0) = -1$.

The correct option is (D)

74. We have $y' = \frac{a}{x} + 2bx + 1$ and $y'(-1) = 0$ and $y'(2) = 0$

$\therefore -a - 2b + 1 = 0$ and $a/2 + 4b + 1 = 0$

$\Rightarrow \frac{a}{2} - 2a + 2 + 1 = 0$

Hence, $a = 2$ and $b = -1/2$.

The correct option is (B)

75. Let $h(x) = f(x) - 2g(x), x \in [0, 1]$

$\Rightarrow \quad h'(x) = f'(x) - 2g'(x)$

Also, $h(0) = f(0) - 2g(0) = 2 - 2 \times 0 = 2$

$h(1) = f(1) - 2g(1) = 6 - 2 \times 2 = 2$

$\therefore \quad h(0) = h(1)$

Since $f(x)$ and $g(x)$ are differentiable in [0, 1], $h(x)$ is also differentiable in [0, 1]. Hence, $h(x)$ is also continuous in [0, 1].

So, all the conditions of Rolle's theorem are satisfied. Hence, there exists a point c, $0 < c < 1$ for which $h'(c) = 0$.

$\therefore \quad f'(c) - 2g'(c) = 0$, i.e., $f'(c) = 2g'(c)$

The correct option is (B)

76. The given function is periodic with period 2π. So the difference between the greatest and least values of the function is the difference between these values on the interval $[0, 2\pi]$.

We have, $f'(x) = -(\sin x + \sin 2x - \sin 3x)$

$$= -4 \sin x \sin \frac{3x}{2} \sin \frac{x}{2}.$$

Hence, $x = 0, \frac{2\pi}{3}, \pi$ and 2π are the critical points. Also

$$f(0) = \frac{7}{6},\ f\left(\frac{2\pi}{3}\right) = \frac{-13}{12},\ f(\pi) = \frac{-1}{6} \text{ and } f(2\pi) = \frac{7}{6}.$$

Hence, the greatest value is $\frac{7}{6}$ and the least value is $\frac{-13}{12}$.

Thus, the difference is

$$\frac{7}{6} - \left(\frac{-13}{12}\right) = \frac{27}{12} = \frac{9}{4}.$$

The correct option is (C)

77. Since $f'(4) = f''(4) = 0$, therefore,

$f(x) = (x - 4)^n + k$, where $n \geq 3$

But f has minimum at $x = 4$, so $n = 4$.

$\therefore \quad f(x) = (x - 4)^4 + k$.

Since $f(4) = 10$, therefore, $k = 10$

Thus, $f(x) = (x - 4)^4 + 10$

The correct option is (B)

78. We have,

$f(x) = (k^2 - 7k + 12) \cos x + 2(k - 4)x + \log 2$

$\quad = (k - 3)(k - 4) \cos x + 2(k - 4)x + \log 2$

$\Rightarrow \quad f'(x) = -(k-3)(k-4)\sin x + 2(k-4)$

$= (k-4)[-(k-3)\sin x + 2]$

Since $f(x)$ does not have critical points, therefore $f'(x) = 0$ does not have any solution in R

$\Rightarrow \quad k \neq 4$ and $2 - (k-3)\sin x = 0$ is not solvable in R

$\Rightarrow \quad k \neq 4$ and $\sin x = \dfrac{2}{k-3}$ is not solvable in R

$\Rightarrow \quad k \neq 4$ and $\left|\dfrac{2}{k-3}\right| > 1$

$\Rightarrow \quad k \neq 4$ and $|k-3| < 2$

$\Rightarrow \quad k \neq 4$ and $-2 < k-3 < 2$

$\Rightarrow \quad k \neq 4$ and $1 < k < 5 \quad \Rightarrow \quad k \in (1, 5) - \{4\}$

The correct option is (B)

79. Since $f(x)$ has a relative minimum at $x = 0$

$\therefore \quad f'(0) = 0$ and $f''(0) > 0$

Now, $\quad y = f(x) + ax + b$

$\Rightarrow \quad \dfrac{dy}{dx} = f'(x) + a$

$\therefore \quad \left.\dfrac{dy}{dx}\right]_{x=0} = f'(0) + a = a = 0$, if $a = 0$

Also, $\quad \dfrac{d^2y}{dx^2} = f''(x)$.

$\therefore \quad \left.\dfrac{d^2y}{dx^2}\right]_{x=0} = f''(0) > 0.$

$\therefore$ y has a relative minimum at $x = 0$, if $a = 0$ and for all b

The correct option is (D)

80. Since $f(x_0) = g(x_0)$ and $f'(x) > g'(x), \forall\, x > x_0$

$\therefore \quad f'(x) - g'(x) > 0$

$\Rightarrow \quad f(x) - g(x)$ is an increasing function $\forall\, x > x_0$.

$\therefore \quad f(x) - g(x) > f(x_0) - g(x_0) = 0$

$\therefore \quad f(x) > g(x), \forall\, x > x_0$.

The correct option is (C)

81. Let $f(x) = ax^2 + bx + c$.

Then, $f(\alpha) = 0 = f(\beta)$.

Also, $f(x)$ is continuous and differentiable in $[\alpha, \beta]$ as it is a polynomial function of x.

Hence, by Rolle's theorem, there exists a k in (α, β), such that

$f'(k) = 0 \quad \Rightarrow \quad 2ak + b = 0 \quad \Rightarrow \quad k = -\dfrac{b}{2a}$.

$\therefore \quad \alpha < -\dfrac{\boldsymbol{b}}{\boldsymbol{2a}} < \beta.$

The correct option is (C)

82. We have, $p'(1) = \lim\limits_{h\to 0} \dfrac{p(1+h) - p(1)}{h}$

$\Rightarrow \quad |p'(1)| = \left|\lim\limits_{h\to 0} \dfrac{p(1+h) - p(1)}{h}\right|$

$$= \lim_{h\to 0} \frac{|p(1+h) - p(1)|}{|h|} \leq \lim_{h\to 0} \frac{|p(1+h) + p(1)|}{|h|} \qquad (1)$$

Now, $|p(x)| \leq \left|e^{x-1} - 1\right|$

$\Rightarrow \quad |p(1+h)| \leq |e^h - 1|$ and $|p(1)| \leq 0 \quad \Rightarrow \quad p(1) = 0$

Hence, $|p'(1)| \leq \lim\limits_{h\to 0} \dfrac{|e^h - 1|}{|h|} = 1$

$\Rightarrow \quad |p'(1)| \leq 1 \quad \Rightarrow \quad |a_1 + 2a_2 + 3a_3 + \ldots + na_n| \leq 1.$

The correct option is (A)

83. $\dfrac{c^4}{r^2} = a^2 \operatorname{cosec}^2 t + b^2 \sec^2 t$

$\Rightarrow \quad r^2 = \dfrac{c^4}{a^2 \operatorname{cosec}^2 t + b^2 \sec^2 t}$

$\Rightarrow \quad r^2 = \dfrac{c^2}{a^2(1 + \cot^2 t) + b^2(1 + \tan^2 t)}$

$\Rightarrow \quad r^2 = \dfrac{c^4}{(a^2 + b^2) + (a^2\cot^2 t + b^2 \tan^2 t)}$

$\Rightarrow \quad r^2 = \dfrac{c^4}{(a+b)^2 + (a\cot t - b\tan t)^2}$

Now, r^2 is maximum if denominator is minimum,

$\therefore$ c is constant

The denominator is minimum if $(a\cot t - b\tan t)^2 = 0$

$\therefore$ Maximum value of $r^2 = \dfrac{c^4}{(a+b)^2}$

$\Rightarrow \quad$ Maximum value of $r = \dfrac{c^2}{a+b}$

The correct option is (C)

84. Function f is decreasing before 1 and increasing after 1. f has least value at $x = 1$ if $\lim\limits_{x\to 1^-} f(x) \geq f(1)$

$\Rightarrow \quad -1 + \log_2 b \geq 3 \quad \Rightarrow \quad b \in [16, \infty)$

The correct option is (C)

85. The first derivative $f'(x)$ exists for all x in [0, 1] which implies that $f(x)$ is continuous for all x in [0, 1]. Also, it is given that $f(0) = f(1)$

Thus, applying Rolle's theorem on $f(x)$ in the interval [0, 1], we have

$f'(C) = 0$ for some c in [0, 1]

The second derivative $f''(x)$ exists for all x in [0, 1] which implies that $f'(x)$ is continuous for all x in [0, 1]

Thus, applying Lagrange's theorem on $f'(x)$ in the interval $[c, x]$ $(c < x \geq 1)$, we have

$f''(\theta_1) = \dfrac{f'(x) - f'(c)}{x - c}$ for some $\theta_1 \in [c, x]$

i.e., $\quad f'(x) = (x-c) f''(\theta_1) \qquad [\therefore \quad f'(c) = 0]$

i.e., $\quad |f'(x)| = |x - c|\,|f''(\theta_1)| < 1$

$[\therefore /x - c/ < 1$ and $|f''(x)| \leq 1 \;\; \triangledown x]$

Similarly, applying Lagrange's theorem on $f'(x)$ in the interval $[x, c]$ $(0 \leq x < c)$, we have

$$f''(\theta_2) = \frac{f'(c) - f'(x)}{c - x} \text{ for some } \theta_2 \text{ in } [x, c]$$

i.e., $f'(x) = (x - c) f''(\theta_2)$

i.e., $|f'(x)| = |x - c|\,|f''(\theta_2)| < 1$

Hence, we have

$|f'(x)| < 1$ for all x in $[0, 1]$

The correct option is (A)

86. $P'''(x)$ is identically equal to 0 implies $P(x)$ is a second degree polynomial.

Let $P(x) = ax^2 + bx + c$. According to the given condition, we have

$P(3) = 9$

i.e., $9a + 3b + c = 9$ (1)

Now, $f(x)$ is continuous at $x = 2$

$$\Rightarrow \lim_{x\to 2} \frac{P(x)}{x - 2} = 7$$

For the above limit to exist, we have

$P(2) = 0$

i.e., $4a + 2b + c = 0$ (2)

Now, applying L' Hospital's rule, we have

$$\lim_{x\to 2} \frac{P(x)}{x - 2}\left(\frac{0}{0}\right) = \lim_{x\to 2} \frac{P'(x)}{1} = P'(2) = 7$$

i.e., $4a + b = 7$ $[P'(x) = 2ax + b]$ (3)

Solving (1), (2) and (3), we have

$a = 2, b = -1, c = -6$

and, $P(x) = 2x^2 - x - 6$

The correct option is (A)

87. Let $f(x) = x^5 - 3x - 1, x \in [1, 2]$

Then, $f'(x) = 5x^4 - 3 > 0\ \forall\, x \in (1, 2)$

$\Rightarrow$ $f(x)$ strictly increases in $(1, 2)$

Also, we have

$f(1) = 1 - 3 - 1 = -3$

and, $f(2) = 32 - 6 - 1 = 25$

Therefore, the curve $y = f(x)$ will cut the X-axis exactly once in $[1, 2]$

i.e., $f(x)$ will vanish exactly once in $[1, 2]$

The correct option is (D)

88. Consider the function

$$f(x) = x - \sin x - k, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Then, $f'(x) = 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right) \geq 0, \forall\, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow$ $f(x)$ increases in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Also, we have

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 1 - a \text{ and } f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 1 - a$$

The curve $y = f(x)$ will cut the x-axis exactly once, if $f\left(-\frac{\pi}{2}\right)$ is negative or zero and $f\left(\frac{\pi}{2}\right)$ is posiive or zero.

i.e., $-\frac{\pi}{2} + 1 - a \leq 0$ and $\frac{\pi}{2} - 1 - a \geq 0$

i.e., $a \geq -\frac{\pi}{2} + 1$ and $a \leq \frac{\pi}{2} - 1$

Hence, we have $a \in \left[1 - \frac{\pi}{2}, \frac{\pi}{2} - 1\right]$

The correct option is (B)

89. Let $f(x) = \frac{x}{x^2 + 10}, x > 0$

Then, $f'(x) = \frac{(x^2 + 10) - 2x^2}{(x^2 + 10)^2}$

$= -\frac{(x + \sqrt{10})(x - \sqrt{10})}{(x^2 + 10)^2} > \triangledown\ 0,\ 0 < x < \sqrt{10}$

$< 0\ \forall\, x > \sqrt{10}$

$\Rightarrow$ $f(x)$ strictly increases in $(0, \sqrt{10})$

and strictly decreases in $(\sqrt{10}, \infty)$

$\Rightarrow$ $f(x)$ has greatest value at $x = \sqrt{10}$

Hence, the given sequence has greatest value at $n = 3$ or, $n = 4$

Now, we have,

$a_3 = \frac{3}{19}$ and $a_4 = \frac{4}{26}$

Hence, $a_3 = \frac{3}{19}$ is the largest term of the given sequence.

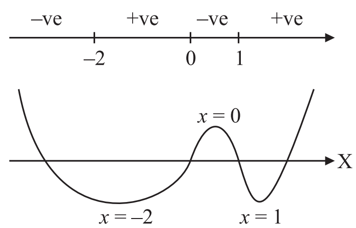
The correct option is (B)

90. Let $f(x) = 3x^4 + 4x^3 - 12x^2 + a$

$\Rightarrow$ $f'(x) = 12(x^3 + x^2 - 2x) = 12x(x - 1)(x + 2)$

From the shape of the curve given below, we have

for four real and district roots, the two minima must lie below the X-axis and the maxima must lie above the X-axis



Thus, we have,

$f(-2) < 0$

i.e., $48 - 32 - 48 + a < 0$ i.e., $a < 32$ (1)

and, $f(1) < 0$

i.e., $3 + 4 - 12 + a < 0$ i.e., $a < 5$ (2)

and, $f(0) > 0$ i.e., $a > 0$ (3)

From (1), (2) and (3), we get $a \in (0, 5)$

The correct option is (A)

91. We have,

$\Rightarrow \quad f(x) = f(x) + f(1 - x)$

and, $\quad f'(x) = f'(x) - f'(1 - x)$

which vanishes at the points given by

$x = 1 - x \quad \Rightarrow \quad x = \dfrac{1}{2}$

Now, we have,

$f''(x) = f''(x) + f''(1 - x)$

$\therefore \quad f''\left(\dfrac{1}{2}\right) = f''\left(\dfrac{1}{2}\right) + f''\left(\dfrac{1}{2}\right) < 0$

[$\therefore \quad f''(x) < 0$ in $(-1, 1)$]

$\Rightarrow \quad$ maxima at $x = \dfrac{1}{2}$

Hence, $f(x)$ strictly increases in $\left(0, \dfrac{1}{2}\right)$

The correct option is (A)

92. Let $f(x) = ax^3 + bx^2 + cx + d$

Then, $f'(x) = 3ax^2 + 2bx + c$

and, $\quad f''(x) = 6ax + 2b$

Now, using the given conditions, we have

$f''(1) = 0 \quad$ i.e., $3a + b = 0$ (1)

$f''(-1) = 0 \quad$ i.e., $3a - 2b + c = 0$ (2)

$f(1) = -6 \quad$ i.e., $a + b + c + d = -6$ (3)

and, $\quad f(-1) = 10$ i.e., $-a + b - c + d = 10$ (4)

Solving the above equations (1) – (4), we have

$a = 1, b = -3, c = -9$ and $d = 5$

Hence, we have,

$f(x) = x^3 - 3x^2 - 9x + 5$

The correct option is (B)

93. We have,

$f(x) = -x^3 + \phi(b), 0 \le x < 1$

$\quad = 2x - 3, \quad 1 \le x \le 3$

where, $\phi(x) = \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}$ (b is a constant)

and, $\quad f'(x) = -3x^2, 0 < x < 1$

$\quad = 2, 1 < x < 3$

$\Rightarrow \quad f(x)$ strictly decreases in (0, 1)

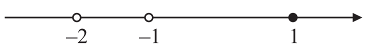
and, strictly increases in (1, 3)

Now, we have for minima at $x = 1$

$f(1) \le f(1^-)$

i.e., $\quad 2 - 3 \le -1 + \phi(b)$ i.e., $\phi(b) \ge 0$

i.e., $\quad \dfrac{(b^2 + 1)(b - 1)}{(b + 1)(b + 2)} \ge 0 \quad \Rightarrow \quad b \in (-2, -1) \cup [1, \infty)$



The correct option is (B)

94. We have,

$f(x) = \dfrac{|x + 1|}{x^2}$

$\Rightarrow \quad f(x) = \begin{cases} \dfrac{-(x + 1)}{x^2} = \dfrac{-1}{x} - \dfrac{1}{x^2}, & x < -1 \\ \dfrac{x + 1}{x^2} = \dfrac{1}{x} + \dfrac{1}{x^2}, & x \ge -1 \end{cases}$

$\therefore \quad f'(x) = \begin{cases} \dfrac{x(x + 2)}{x^4} & > 0 \ \forall \ x \in (-\infty, -2) \\ & < 0 \ \forall \ x \in (-2, -1) \\ \dfrac{-x(x + 2)}{x^4} & > 0 \ \forall \ x \in (-1, 0) \\ & < 0 \ \forall \ x \in (0, \infty) \end{cases}$

$\Rightarrow \quad f$ is decreasing in $(-2, -1) \cup (0, \infty)$

The correct option is (C)

95. Let $f'(x) = ax^2 + bx + c$

Integrating both sides,

$\Rightarrow \quad f(x) = \dfrac{ax^3}{3} + \dfrac{bx^2}{2} + cx + d$

$\Rightarrow \quad f(0) = d$ and $f(1) = \dfrac{a}{3} + \dfrac{b}{2} + c + d$

Since, Rolle's theorem is applicable

$\Rightarrow \quad f(0) = f(1) \quad \Rightarrow \quad d = \dfrac{a}{3} + \dfrac{b}{2} + c + d$

$\Rightarrow \quad 2a + 3b + 6c = 0$

Hence, required condition is $2a + 3b + 6c = 0$

The correct option is (A)

96. Let, $f(x) = x^3 - 3x + a$

$\Rightarrow \quad f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$

Now, $f(1) = a - 2, f(-1) = a + 2$

The roots would be real and distinct if,

$f(1) f(-1) < 0$

$\Rightarrow \quad (a - 2)(a + 2) < 0 \quad$ or

$\Rightarrow \quad -2 < a < 2$

Thus the given equation would have real and distinct roots if $a \in (-2, 2)$

The correct option is (D)

97. Let (α, β) be the point of intersection of given curves,

Then, $\dfrac{\alpha^2}{a} + \dfrac{\beta^2}{b} = 1$ (1)

and, $\dfrac{\alpha^2}{a_1} + \dfrac{\beta^2}{b_1} = 1$ (2)

Differentiating $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with respect to x we get,

$\dfrac{2x}{a} + \dfrac{2y}{b} \cdot \dfrac{dy}{dx} = 0 \Rightarrow \left(\dfrac{dy}{dx}\right) = \dfrac{-bx}{ay}$

$\therefore \quad \left(\frac{dy}{dx}\right)_{(\alpha,\beta)} = \frac{-b\alpha}{a\beta} = m_1 \quad \text{(say)} \qquad (3)$

Similarly, from $\frac{x^2}{a_1} + \frac{y^2}{b_1} = 1$, we get

$$\left(\frac{dy}{dx}\right)_{(\alpha,\beta)} = \frac{-b_1\alpha}{a_1\beta} = m_2 \quad \text{(say)} \qquad (4)$$

Since the curves cut each other orthogonally, we have $m_1 m_2 = -1$

$$\Rightarrow \frac{-b\alpha}{a\beta}\cdot\frac{-b_1\alpha}{a_1\beta} = -1$$

$$\Rightarrow bb_1\alpha^2 + aa_1\beta^2 = 0 \qquad (5)$$

Subtracting (1) from (2), we get,

$$\alpha^2\left(\frac{1}{a_1} - \frac{1}{a}\right) + \beta^2\left(\frac{1}{b_1} - \frac{1}{b}\right) = 0 \qquad (6)$$

From (5) and (6),

$$\frac{bb_1}{\left(\frac{1}{a} - \frac{1}{a_1}\right)} = \frac{aa_1}{\left(\frac{1}{b} - \frac{1}{b_1}\right)}$$

or, $bb_1\left(\frac{1}{b} - \frac{1}{b_1}\right) = aa_1\left(\frac{1}{a} - \frac{1}{a_1}\right)$

i.e., $b_1 - b = a_1 - a$

Therefore, $a - b = a_1 - b_1$

The correct option is (B)

98. Given x and y are sides of two squares, thus the area of two squares are x^2 and y^2

Now, $\frac{d(y^2)}{d(x^2)} = \frac{2y\frac{dy}{dx}}{2x} = \frac{y}{x}\frac{dy}{dx} \qquad (1)$

The given curve is,

$y = x - x^2$

$$\Rightarrow \frac{dy}{dx} = 1 - 2x \qquad (2)$$

Thus, $\frac{d(y^2)}{d(x^2)} = \frac{y}{x}(1 - 2x)$ [from (1) and (2)]

or, $\frac{d(y^2)}{d(x^2)} = \frac{(x - x^2)(1 - 2x)}{x}$

$$\Rightarrow \frac{d(y^2)}{d(x^2)} = (2x^2 - 3x + 1)$$

The correct option is (A)

99. Slope of the given line $3x + 2y + 1 = 0$ is $\left(\frac{-3}{2}\right)$. We first find the point on the curve at which the tangent is parallel to the given line. Differentiating the given curve with respect to x, we get,

$$6x - 8y\frac{dy}{dx} = 0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{3x_1}{4y_1} = \frac{-3}{2} \quad \text{[Since parallel to } 3x + 2y = 1]$$

Also, the point (x_1, y_1) lies on $3x^2 - 4y^2 = 72$, therefore,

$\Rightarrow 3x_1^2 - 4y_1^2 = 72$

$$\Rightarrow \frac{3x_1^2}{y_1^2} - 4 = \frac{72}{y_1^2}$$

$$\Rightarrow 3(4) - 4 = \frac{72}{y_1^2} \quad \left[as\ \frac{x_1}{y_1} = -2\right]$$

$\Rightarrow y_1^2 = 9$ or $y_1 = \pm 3$

Required points are $(-6, 3)$ and $(6, -3)$

Distance of $(-6, 3)$ from the given line,

$$= \frac{|-18 + 6 + 1|}{\sqrt{13}} = \frac{11}{\sqrt{13}}$$

and distance of (6, 3) from the given line,

$$= \frac{|18 - 6 + 1|}{\sqrt{13}} = \frac{13}{\sqrt{13}} = \sqrt{13}$$

Thus, $(-6, 3)$ is the required point

The correct option is (C)

100. Given $f(x) = (a^2 - 3a + 3)\cos\frac{x}{2} + (a - 1)x$

$$\Rightarrow f'(x) = \frac{-1}{2}(a - 1)(a - 2)\sin\left(\frac{x}{2}\right) + (a - 1)$$

$$\Rightarrow f'(x) = (a - 1)\left[1 - \frac{1}{2}(a - 2)\sin\left(\frac{x}{2}\right)\right]$$

If $f(x)$ possesses critical points, then

$f'(x) = 0$

$$\Rightarrow (a - 1)\left[1 - \left(\frac{a - 2}{2}\right)\sin\frac{x}{2}\right] = 0$$

$$\Rightarrow a = 1 \text{ and } 1 - \left(\frac{a - 2}{2}\right)\sin\frac{x}{2} = 0$$

$$\Rightarrow a = 1 \text{ and } \sin\left(\frac{x}{2}\right) = \frac{2}{a - 2}$$

$$\Rightarrow a = 1 \text{ and } \left|\frac{2}{a - 2}\right| \le 1$$

$\Rightarrow a = 1$ and $|a - 2| \ge 2$

$\Rightarrow a = 1$ and $a - 2 \ge 2$ or $a - 2 \le -2$

$\Rightarrow a = 1$ and $a \ge 4$ or $a \le 0$

$\Rightarrow a = 1$ and $a \in (-\infty, 0] \cup [4, \infty)$

Therefore, $a \in (-\infty, 0] \cup \{1\} \cup [4, \infty)$

The correct option is (C)

101. We have,

$$f(x) = \int_0^x |\log_2(\log_3(\log_4(\cos t + a)))|\,dt$$

Differentiating with respect to x, we get

$f'(x) = |\log_2(\log_3(\log_4(\cos x + a)))|$,

which is clearly increasing for all $x \in R$

But it must be defined,

i.e., $\log_3 \log_4(\cos x + a) \geq 0, \forall x \in R$

$\Rightarrow \log_4(\cos x + a) \geq 1, \forall x \in R$

$\Rightarrow \cos x + a \geq 4, \forall x \in R$

$\Rightarrow a \geq 4 - \cos x$

$\Rightarrow a \geq 5$

Thus, $f(x)$ is increasing for all real values of x when $a \geq 5$

The correct option is (B)

102. We have,

$f(x) = (x^2 - 4)^n (x^2 - x + 1)$ assumes local minima at $x = 2$

$\Rightarrow f(2) < f(2 - h)$

and, $f(2) < f(2 + h)$,

where $h > 0$. Since $f(2) = 0$

$\Rightarrow f(2 - h) > 0$

and, $f(2 + h) > 0, \forall h > 0$

$\Rightarrow (-h)^n (4 - h)^n [h^2 - 3h + 1] > 0$

and, $h^n(4 + h)^n (h^2 + 5h + 1) > 0$

i.e., $(-h)^n > 0$ $\quad [\because (4 - h) > 0, h^2 - 3h + 1 > 0,$ $4 + h > 0, h^2 + 5h + 1 > 0 \ \forall h > 0]$

$\Rightarrow$ n is an even number

The correct option is (A)

103. Since the function $f(x)$ increases for all x, therefore,

$$f'(x) = \left(1 - \frac{\sqrt{21 - 4b - b^2}}{b + 1}\right) 3x^2 + 5 > 0, \forall x \in R$$

$$\Rightarrow \left(1 - \frac{\sqrt{21 - 4b - b^2}}{b + 1}\right) > 0$$

and, $$(0)^2 - 4 \times 3 \left(\frac{1 - \sqrt{21 - 4b - b^2}}{b + 1}\right) 5 < 0$$

$[\because ax^2 + bx + c > 0 \ \forall x \in R \Rightarrow a > 0 \text{ and } D > 0]$

$$\Rightarrow 1 - \frac{\sqrt{21 - 4b - b^2}}{b + 1} > 0$$

The above inequality holds, when

(i) $b + 1 < 0$ and

(ii) $21 - 4b - b^2 > 0$

$\therefore b < -1$ and $b^2 + 4b - 21 < 0$

$\Rightarrow b < -1$ and $(b + 7)(b - 3) < 0$

$\Rightarrow b < -1$ and $-7 < b < 3$

$\therefore b \in (-7, -1)$ (1)

Again, when $b + 1 > 0$, $f(x)$ will be increasing for all x,

if, $21 - 4b - b^2 > 0$ and $1 > \dfrac{\sqrt{21 - 4b - b^2}}{b + 1}$

or, $b^2 + 4b - 21 < 0$

and, $(b + 1) > (21 - 4b - b^2)$ [as $b + 1 > 0$]

or, $(b + 7)(b - 3) < 0$ and $b^2 + 3b - 10 > 0$

$\Rightarrow (-7 < b < 3)$ and ($b < -5$ or $b > 2$)

$\Rightarrow 2 < b < 3$ (2)

From (1) and (2), we have

$b \in (-7, -1) \cup (2, 3)$

The correct option is (C)

104. $$f(x) = \int_0^x (bt^2 + b + \cos t)\, dt$$

$\Rightarrow f'(x) = bx^2 + b + \cos x$

Case I: $f'(x) \geq 0 \ \forall x \in R$

$\Rightarrow bx^2 + b + \cos x \geq 0, \forall x \in R$

$\Rightarrow bx^2 + b - 1 \geq 0, \forall x \in R$

$\Rightarrow b > 0$ and $(-4b(b - 1)) \leq 0$

$\Rightarrow b \geq 1$ (1)

Case II: $f'(x) \leq 0, \forall x \in R$

$\Rightarrow bx^2 + b + 1 \leq 0, \forall x \in R$

$\Rightarrow b < 0$ and $(0 - 4b(b + 1)) \leq 0$

$\Rightarrow b + 1 \leq 0$

$\Rightarrow b \leq -1$ (2)

From (1) and (2), we get $b \in (-\infty, -1] \cup [1, \infty)$.

The correct option is (B)

105. We have,

$f(x) = (x - 3)(x - 4)(x - 5)(x - 6)$

Consider the interval [3, 4]

Since $f(x)$ is a polynomial function,

$\therefore$ it is continuous in [3, 4] and derivable in (3, 4)

Also, $f(3) = f(4) = 0$

$\therefore$ By Rolle's theorem, there exists a point $x = c$ in (3, 4) such that $f'(c) = 0$ i.e., there exists a root of $f'(x) = 0$ in the open interval (3, 4). Clearly, this root is positive. Similarly, two more positive roots of $f'(x) = 0$ exist in the other intervals (4, 5) and (5, 6). Hence, three roots of $f'(x) = 0$ are positive and they lie in $(3, 4) \cup (4, 5) \cup (5, 6)$.

The correct option is (B)

106. $f'(x) = \sin a \tan^2 x \sec^2 x + (\sin a - 1)\sec^2 x$

$= (\sin a \tan^2 x + \sin a - 1)\sec^2 x$

At critical points, we must have $f'(x) = 0$

$\Rightarrow \sin a \tan^2 x + \sin a - 1 = 0 \quad (\because \sec^2 x \neq 0$ for any $x \in R)$

$$\Rightarrow \tan^2 x = \frac{1 - \sin a}{\sin a}$$

Since $a \in [\theta, 2\theta]$, $\dfrac{1 - \sin a}{\sin a} < 0$

$\therefore \tan^2 x = \dfrac{1 - \sin a}{\sin a}$ has no solution in R

$\Rightarrow f(x)$ has no critical points.

The correct option is (D)

107. $f''(x) > 0 \ \forall x \in R \Rightarrow f'(x)$ is increasing $\forall x \in R$

Now, $g'(x) = -f'(2 - x) + f'(4 + x)$

If $g'(x) > 0$, then $f'(4 + x) > f'(2 - x)$

$\Rightarrow 4 + x > 2 - x$ $\quad [\because f'(x)$ is increasing $\forall x \in R]$

$\Rightarrow \quad 2x > -2 \quad \text{or} \quad x > -1$

$\therefore \quad g(x)$ is increasing in $(-1, \infty)$.

The correct option is (C)

108. Curves will intersect if $x^2 - 16x + c = 0$ has real roots.

$\therefore \quad 256 - 4c \geq 0$ i.e., $c \leq 64$

$x^2 - 4y^2 + c = 0$

$$\Rightarrow \quad 2x - 8y\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{x}{4y}$$

Again, $\quad y^2 = 4x$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{y}$$

Since the curves intersect orthoganally,

$$\therefore \quad \frac{x}{4y}\cdot\frac{2}{y} = -1$$

$\Rightarrow \quad x = -2y^2 = -2(4x) \qquad (\therefore \quad y^2 = 4x)$

$\Rightarrow \quad 9x = 0 \Rightarrow x = 0, \Delta y = 0$

But if $y = 0$, slopes of both curves are undefined

$\therefore$ Curves cannot cut orthogonally.

The correct option is (D)

109. $f'(x) = 2x - \dfrac{\lambda}{x^2}$

$$\therefore \quad f'(x) = 0 \quad \Rightarrow \quad x = \left(\frac{\lambda}{2}\right)^{1/3}$$

If $\quad \lambda = 16, x = 2$

Now, $\quad f''(x) = 2 + \dfrac{2\lambda}{x^3}$

$\therefore \quad$ If $\lambda = 16, f''(x) > 0$

i.e., $f(x)$ has a minimum at $x = 2$

$$\text{Also, } f''\left(\left(\frac{\lambda}{2}\right)^{1/3}\right) = 2 + \frac{2\lambda}{\lambda/2} = 2 + 4 > 0$$

Hence, $f(x)$ is maximum for no real value of λ.

When $\lambda = -1, f''(x) = 0$ if $x = 1$.

$\therefore \quad f(x)$ has a point of inflexion at $x = 1$.

The correct option is (B)

110. Let $y = f(x) = ax^2 + bx + c$

$\therefore f'(x) = 2ax + b$

$f(0) = c$ and $f'(0) = b$

$f'(x)_{\{\text{at }(1,\,1)\}} = 2a + b = 1 \qquad (1)$

$f(1) = a + b + c = 1 \qquad (2)$

Subtracting (2) from (1), we have $a - c = 0$ or $a = c$

Now, $\quad 2f(0) + f'(0) = 2c + b = 2a + b = 1$

The correct option is (B)

111. Let $P(x, y)$ be a point on the curve

$\ln(x^2 + y^2) = c\tan^{-1}\dfrac{y}{x}$

Differentiating both sides with respect to x, we get

$$\frac{2x + 2yy'}{(x^2 + y^2)} = \frac{c(xy' - y)}{(x^2 + y^2)}$$

$$\Rightarrow \quad y' = \frac{2x + cy}{cx - 2y} = m_1 \text{ (say)}$$

and, $\quad$ slope of $OP = \dfrac{y}{x} = m_2$ (say)

Let the angle between the tangent at P and OP be θ

$$\text{Then, } \tan\theta = \left|\frac{m_1 + m_2}{1 + m_1m_2}\right|$$

$$= \left|\frac{\dfrac{2x + cy}{cx - 2y} - \dfrac{y}{x}}{1 + \dfrac{2xy + cy^2}{cx^2 - 2xy}}\right| = \frac{2}{c}$$

$\therefore \quad \theta = \tan^{-1}\left(\dfrac{2}{c}\right)$, which is independent of x and y.

The correct option is (C)

112. Consider $f(x) = e^{x/3}(ax^2 + bx + c)$. Clearly, $f(x) = 0$ has two distinct positive roots α and β.

So, by Rolle's theorem, $f'(x) = 0$ has at least one root in (α, β)

or, $\dfrac{1}{3}e^{3x}\left\{ax^2 + (b + 6a)x + c + 3b\right\} = 0$ has at least one root in (α, β)

$\Rightarrow \quad ax^2 + (b + 6a)x + c + 3b = 0$ has at least one positive root.

The correct option is (C)

113. Define a function $h(x) = f(x) - f(a) + k(x^3 - a^3)$, where k is chosen in such a way that $h(b) = 0$. Also, $h(a) = 0$. So, the function $h(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists $c \in (a, b)$ such that $h'(c) = 0$.

$\Rightarrow \quad f'(c) + k(3c^2) = 0$

$$\Rightarrow \quad f'(C) = \frac{f(b) - f(a)}{b^3 - a^3}(3c^2)$$

$$\left[\text{Where } k = \frac{-(f(b) - f(a))}{b^3 - a^3}\right]$$

The correct option is (B)

114. Let $\phi(x) = x^2 ln(4) - 16\, ln(x)$, which is continuous on [4, 5] and differentiable on (4, 5), so by LMVT,

$$\frac{\phi(5) - \phi(4)}{5 - 4} = f'(c),\ c \in (4, 5)$$

$$\text{Now, } \phi(5) - \phi(4) = \ln\left(\frac{4^{25}}{5^{16}}\right)$$

$$\text{and, } \phi'(C) = \frac{2}{c}(c^2 \ln 4 - 8)$$

$$\Rightarrow \quad c\ln\left(\frac{4^{25}}{5^{16}}\right) = 2(c^2 \ln 4 - 8)$$

The correct option is (B)

More than One Option Correct Type

115. Let $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

Then, $f'(x) = \dfrac{x\cos x - \sin x}{x^2}$

$= \dfrac{\cos x(x - \tan x)}{x^2} < 0$ if $x \in \left(0, \dfrac{\pi}{2}\right)$.

$\left(\because \tan x > x \text{ and } \cos x > 0 \text{ when } 0 < x < \dfrac{\pi}{2}\right)$

$\therefore$ $f(x)$ is decreasing in . $\left(0, \dfrac{\pi}{2}\right)$ Since $0 < x < \dfrac{\pi}{2}$,

$\therefore$ $f\left(\dfrac{\pi}{2}\right) < f(x) < f(0) \Rightarrow \dfrac{2}{\pi} < \dfrac{\sin x}{x} < 1$.

The correct option is (B) and (C)

116. We know that $\sin x < x$ if $0 < x < \dfrac{\pi}{2}$ (1)

Since $\cos x$ is decreasing in $\left(0, \dfrac{\pi}{2}\right)$, $\cos(\sin x) > \cos x$.

Also, since $0 < x < \dfrac{\pi}{2}$, π $0 < \cos x < 1 < \dfrac{\pi}{2}$

π$\sin(\cos x) < \cos x$ [Using (1)]

Hence, $\cos(\sin x) > \cos x > \sin(\cos x)$ if $0 < x < \dfrac{\pi}{2}$.

The correct option is (A) and (C)

117. Let $f(x) = 1 + x^p - (1 + x)^p$

$\Rightarrow f'(x) = px^{p-1} - p(1 + x)^{p-1}$

$= p\left(\dfrac{1}{x^{1-p}} - \dfrac{1}{(1+x)^{1-p}}\right)$

≥ 0 if $p \geq 0$, $1 - p \geq 0$, $x > 0$

$\therefore$ $f(x)$ increases when $x > 0$ and $0 \leq p \leq 1$.

Since $x > 0$, $f(x) > f(0)$

$\therefore$ $1 + x^p - (1 + x)^p > 0$ $(\because f(0) = 0)$

i.e., $1 + x^p > (1 + x)^p$ if $0 \leq p \leq 1$ and $x > 0$

The correct option is (B) and (C)

118. We have,

$f(x) = |x + 2| + |x - 1| = \begin{cases} -2x - 1, & x < -2 \\ 3, & -2 \leq x \leq 1 \\ 2x + 1, & x > 1 \end{cases}$

$\therefore$ $f'(x) = \begin{cases} -2, & x < -2 \\ \text{does not exist}, & x = -2 \\ 0, & -2 < x < 1 \\ \text{does not exist}, & x = 1 \\ 2, & x > 1 \end{cases}$

Thus, $f(x)$ is increasing in $(1, \infty)$ and decreasing in $(-\infty, -2)$.

The correct option is (A) and (D)

119. We have, $f''(x) < 0$, for $0 \leq x \leq 1$

$\Rightarrow$ $f'(x)$ is a decreasing function in [0, 1].

Now, $g'(x) = f'(x) - f'(1 - x)$.

$g(x)$ is increasing if $g'(x) > 0$

$\Rightarrow f'(x) > f'(1 - x)$

$\Rightarrow x < 1 - x$ ($f'(x)$ is decreasing)

i.e., $x < \dfrac{1}{2}$

$\therefore$ $g(x)$ is an increasing function for $0 < x < \dfrac{1}{2}$.

Also, $g(x)$ is decreasing if $g'(x) < 0$

$\Rightarrow f'(x) < f'(1 - x)$

$\Rightarrow x > 1 - x$ $(\because f'(x)$ is decreasing)

i.e., $x > \dfrac{1}{2}$

$\therefore$ $g(x)$ is a decreasing function for $\dfrac{1}{2} < x < 1$.

The correct option is (B) and (C)

120. $f(x) = \dfrac{|x - 1|}{x^2} = \begin{cases} \dfrac{x - 1}{x^2}, & x \geq 1 \\ \dfrac{1 - x}{x^2}, & x < 1, x \neq 0 \end{cases}$

$\therefore$ $f'(x) = \begin{cases} \dfrac{x - 2}{x^3}, & x < 1, x \neq 0 \\ \text{does not exist}, & x = 1 \\ \dfrac{2 - x}{x^3}, & x > 1 \end{cases}$

Clearly, $f'(x) > 0$ for $x < 0$ or $1 < x < 2$

and $f'(x) < 0$ for $0 < x < 1$ or $x > 2$.

Thus, $f(x)$ is increasing for $(-\infty, 0) \cup (1, 2)$ and decreasing for $(0, 1) \cup (2, \infty)$.

The correct option is (A) and (C)

121. We have,

$h'(x) = f'(x)\,[1 - 2f(x) + 3(f(x))^2]$

$= 3f'(x)\left[(f(x))^2 - \dfrac{2}{3}f(x) + \dfrac{1}{3}\right]$

$= 3f'(x)\left[\left(f(x) - \dfrac{1}{3}\right)^2 + \dfrac{2}{9}\right]$

Note that $h'(x) < 0$ whenever $f'(x) < 0$ and $h'(x) > 0$ whenever $f'(x) > 0$. Thus, $h(x)$ increases (decreases) whenever $f(x)$ increases (decreases).

The correct option is (A) and (C)

122. Let $h(x) = f(x) - g(x)$

Since, $f'(x) > g'(x)\ \forall x \in R$, therefore,

$h'(x) = f'(x) - g'(x) > 0\ \forall x \in R$

$\Rightarrow$ $h(x)$ is an increasing function $\forall x \in R$

But $h(0) = f(0) - g(0) = 0$,

So, for $x > 0$, we must have $h(x) > h(0) = 0$

and for $x < 0$, we have $h(x) < h(0) = 0$.

$\Rightarrow$ $f(x) > g(x)\ \forall\ x \in (0, \infty)$

and $f(x) < g(x)\ \forall\ x \in (-\infty, 0)$

The correct option is (A) and (B)

123. We have, $f'(x) = e^{-x^4/4}(4 - x^2)$.

For maximum or minimum, $f'(x) = 0$

$\Rightarrow$ $e^{-x^4/4}(4 - x^2) = 0 \Rightarrow x = -2, 2$

Also, $f''(x) = e^{-x^4/4}(-2x) + e^{-x^4/4} \times (4 - x^2) \times (-x^3)$

$$\begin{cases} < 0 \text{ if } x = 2 \\ \text{and} \\ > 0 \text{ if } x = -2 \end{cases}$$

$\therefore$ $f(x)$ has a maximum at $\boldsymbol{x} = 2$

and minimum at $\boldsymbol{x} = -2$

The correct option is (A) and (B)

124. We have, $f(x) = \sin x + \dfrac{1}{2}\cos 2x$

$\Rightarrow$ $f'(x) = \cos x - \sin 2x = \cos x(1 - 2\sin x)$.

$\therefore$ $f'(x) = 0$

$\Rightarrow$ $\cos x(1 - 2\sin x) = 0 \Rightarrow \cos x = 0$ or $\sin x = \dfrac{1}{2}$

$\Rightarrow$ $x = \dfrac{\pi}{2}$ or $x = \dfrac{\pi}{6}$ $\left(\because 0 \le x \le \dfrac{\pi}{2}\right)$

Now, $f(0) = \dfrac{1}{2}, f\left(\dfrac{\pi}{6}\right) = \dfrac{3}{4}$ and $f\left(\dfrac{\pi}{2}\right) = \dfrac{1}{2}$

Of these values, the minimum value is $\dfrac{1}{2} < \dfrac{\mathbf{4}}{\mathbf{3}} < \dfrac{\mathbf{3}}{\mathbf{2}}$.

The correct option is (A) and (D)

125. Let $f'(x) = (x - a)^{2n}(x - b)^{2m+1}$

The extreme values of f are given by $f'(x) = 0$

$\Rightarrow$ $(x - a)^{2n} \times (x - b)^{2m+1} = 0 \Rightarrow x = a, b$.

For $x < b$, $(x - b)^{2m+1}$ is negative and for $x > b$, $(x - b)^{2m+1}$ is positive ($\because$ $2m + 1$ is odd).

Thus, f' changes sign from negative to positive as x passes through b and so, f has a minimum at $x = b$.

Since $2n$ is an even integer, $(x - a)^{2n}$ does not change sign as x passes through a, i.e., $f'(x)$ does not change sign as x passes through a. Hence, f has neither a maximum nor a minimum at $x = a$.

The correct option is (A) and (C)

126. We have,

$f(x) = |x| + |x - 1| + |x - 2|$

$$= \begin{cases} -3x + 3, & x < 0 \\ -x + 3, & 0 \le x < 1 \\ x + 1, & 1 \le x < 2 \\ 3x - 3, & x \ge 2 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -3 & x < 0 \\ \text{does not exist} & x = 0 \\ -1 & 0 < x < 1 \\ \text{does not exist} & x = 1 \\ 1 & 1 < x < 2 \\ \text{does not exist} & x = 2 \\ 3 & x > 2 \end{cases}$$

Clearly, $f(x)$ has minima at $x = 1$ and neither maxima nor minima at $x = 0$ and $x = 2$.

The correct option is (A), (C) and (D)

127. Let $f(x) = 1 + x^p - (1 + x)^p$

$\Rightarrow$ $f'(x) = px^{p-1} - p(1 + x)^{p-1}$

$= p\left(\dfrac{1}{x^{1-p}} - \dfrac{1}{(1+x)^{1-p}}\right)$

≥ 0 if $p \ge 0$, $1 - p \ge 0$, $x > 0$

$\therefore$ $f(x)$ increases when $x > 0$ and $0 \le p \le 1$.

Since $x > 0$, $f(x) > f(0)$

$\therefore$ $1 + x^p - (1 + x)^p > 0$ ($\because f(0) = 0$)

i.e., $1 + x^p > (1 + x)^p$ if $0 \le p \le 1$ and $x > 0$

The correct option is (B) and (C)

128. We have, $f''(x) < 0$, for $0 \le x \le 1$

$\Rightarrow$ $f'(x)$ is a decreasing function in [0, 1].

Now, $g'(x) = f'(x) - f'(1 - x)$.

$g(x)$ is increasing if $g'(x) > 0$

$\Rightarrow$ $f'(x) > f'(1 - x)$

$\Rightarrow$ $x < 1 - x$ ($\because$ $f'(x)$ is decreasing)

i.e., $x < \dfrac{1}{2}$

$g(x)$ is an increasing function for $0 < x < \dfrac{1}{2}$

Also, $g(x)$ is decreasing if $g'(x) < 0$

$\Rightarrow$ $f'(x) < f'(1 - x)$

$\Rightarrow$ $x > 1 - x$ ($\because$ $f'(x)$ is decreasing)

i.e., $x > \dfrac{1}{2}$

$\therefore$ $g(x)$ is a decreasing function for $\dfrac{1}{2} < \boldsymbol{x} < 1$

The correct option is (B) and (C)

129. $f(x) = \dfrac{|x-1|}{x^2} = \begin{cases} \dfrac{x-1}{x^2}, & x \geq 1 \\ \dfrac{1-x}{x^2}, & x < 1, x \neq 0 \end{cases}$

$\therefore \quad f'(x) = \begin{cases} \dfrac{x-2}{x^3}, & x < 1, x \neq 0 \\ \text{does not exist}, & x = 1 \\ \dfrac{2-x}{x^3}, & x > 1 \end{cases}$

Clearly, $f'(x) > 0$ for $x < 0$ or $1 < x < 2$

and, $f'(x) < 0$ for $0 < x < 1$ or $x > 2$

Thus, $f(x)$ is increasing for $(-\infty, 0) \cup (1, 2)$ and decreasing for $(0, 1) \cup (2, \infty)$

The correct option is (A) and (C)

130. We have,

$h'(x) = f'(x)\,[1 - 2f(x) + 3(f(x))^2]$

$$= 3f'(x)\left[(f(x))^2 - \frac{2}{3}f(x) + \frac{1}{3}\right]$$

$$= 3f'(x)\left[\left(f(x) - \frac{1}{3}\right)^2 + \frac{2}{9}\right]$$

Note that $h'(x) < 0$ whenever $f'(x) < 0$ and $h'(x) > 0$ whenever $f'(x) > 0$. Thus, $h(x)$ increases (decreases) whenever $f(x)$ increases (decreases)

The correct option is (A) and (C)

131. Let $h(x) = f(x) - g(x)$

Since, $f'(x) > g'(x)\ \forall\, x \in R$, therefore,

$h'(x) = f'(x) - g'(x) > 0\ \forall\, x \in R$.

$\Rightarrow$ $h(x)$ is an increasing function $\forall x \in R$

But $h(0) = f(0) - g(0) = 0$,

So, for $x > 0$, we must have $h(x) > h(0) = 0$

and for $x < 0$, we have $h(x) < h(0) = 0$

$\Rightarrow$ $f(x) > g(x)\ \forall\, x \in (0, \infty)$

and, $f(x) < g(x)\ \forall\, x \in (-\infty, 0)$

The correct option is (A) and (B)

132. We have,

$f'(x) > 0$ and $g'(x) < 0\ \forall\, x \in R$

$\therefore$ $f(x)$ is increasing and $g(x)$ is decreasing $\forall\, x \in R$

Hence, we have,

$g(x) > g(x+1)$ and $fog(x) > fog(x+1)$

$g(x) < g(x-1)$ and $fog(x) < fog(x-1)$

$f(x) < f(x+1)$ and $gof(x) > gof(x+1)$

$f(x) > f(x-1)$ and $gof(x) < gof(x-1)$

The correct option is (A) and (C)

133. Let the points at which normal be drawn is (x_1, y_1). Then, it must satisfy $ay^2 = x^3$,

i.e., $a y_1^2 = x_1^3$ or $y_1 = \pm\sqrt{\dfrac{x_1^3}{a}}$

Now, differentiating both sides of the given curve with respect to x we get; $2ay\dfrac{dy}{dx} = 3x^2$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{3x_1^2}{2ay_1} = \frac{3x_1^2}{2a\sqrt{\dfrac{x_1^3}{a}}} = \frac{3}{2}\sqrt{\frac{x_1}{a}} \qquad (1)$$

Thus, slope of normal $\Rightarrow \left(-\dfrac{dx}{dy}\right)_{(x_1, y_1)} = \dfrac{-2}{3}\sqrt{\dfrac{a}{x_1}}$

Since the normal line makes equal intercept with the axes, therefore, slope $= \pm 1$

$$\Rightarrow \frac{-2}{3}\sqrt{\frac{a}{x_1}} = \pm 1 \Rightarrow x_1 = \frac{4a}{9}$$

Hence, the required points are $\left(\dfrac{4a}{9}, \dfrac{8a}{27}\right)$ and $\left(\dfrac{4a}{9}, \dfrac{-8a}{27}\right)$.

The correct option is (B) and (C)

134. We have, $y = 8t^3 - 1$ and $x = 4t^2 + 3$

$\Rightarrow \dfrac{dy}{dt} = 24t^2$ and $\dfrac{dx}{dt} = 8t$

$\therefore \dfrac{dy}{dx} = \dfrac{24t^2}{8t} = 3t$

$\therefore$ tangent at t is

$y - (8t^3 - 1) = 3t(x - 4t^2 - 3) \qquad (1)$

Let the tangent meet the curve again at the point t' (say)

$\therefore \quad (8t'^3 - 1) - (8t^3 - 1) = 3t(4t'^2 + 3 - 4t^2 - 3)$

$\Rightarrow \quad 8(t'^3 - t^3) = 3t\,.\,4(t' - t)(t' + t)$

$\Rightarrow \quad 2(t'^2 + t^2 + tt') = 3t(t' + t)$

$\Rightarrow \quad 2t'^2 - tt' - t^2 = 0$

$\Rightarrow \quad (t' - t)(2t' + t) = 0 \Rightarrow t' = t$ or $\dfrac{-t}{2}$

$\therefore$ the line (1), which is tangent at the point t is normal at the point $\dfrac{-t}{2}$

$\therefore \quad 3t = \dfrac{2}{3t} \Rightarrow t = \pm\dfrac{\sqrt{2}}{3}$

Hence, the required lines are

$$\sqrt{2}x - y = \frac{89\sqrt{2}}{27} + 1 \text{ and } \sqrt{2}x + y = \frac{89\sqrt{2}}{27} - 1$$

The correct option is (B) and (C)

135. $f'(x) = 1$ if $-1 \leq x < 0$

$Lf'(0) = 1,\ Rf'(0) = \dfrac{1}{2}$

$\therefore$ $f'(0)$ does not exist

Also, $f'(x) = \dfrac{1}{2}$ for $0 < x \leq 1$

and, $f'(x) = 1$ for $-1 \leq x < 0$

$\therefore$ $f'(x) \neq 0$ for any $x \in [-1, 1]$

$\therefore$ $f(x)$ has neither a maximum nor a minimum.

The correct option is (A) and (C)

Passage Based Questions

136. For the intersection of the given curves

$|x^2 - 1| = |x^2 - 3| \Rightarrow (x^2 - 1)^2 = (x^2 - 3)^2$

$\Rightarrow (x^2 - 1)^2 - (x^2 - 3)^2 = 0$

$\Rightarrow [(x^2 - 1) - (x^2 - 3)][(x^2 - 1) + (x^2 - 3)] = 0$

$\Rightarrow 2[(2x^2 - 4) = 0 \Rightarrow 2x^2 = 4 \Rightarrow x = \pm\sqrt{2}$

neglecting $x = -\sqrt{2}$ as $x > 0$

We have point of intersection as $x = \sqrt{2}$

$\therefore y = |x^2 - 1| = (x^2 - 1)$ in the neighbourhood of $x = \sqrt{2}$

and $y = -(x^2 - 3)$ in the neighbourhood of $x = \sqrt{2}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{c_1} = 2x = 2\sqrt{2}$

and, $\left(\frac{dy}{dx}\right)_{c_2} = -2x = 2\sqrt{2}$

Hence, if θ is the angle between them, then

$$\tan\theta = \left|\frac{2\sqrt{2} - (-2\sqrt{2})}{1 + 2\sqrt{2}(-2\sqrt{2})}\right| = \left|\frac{4\sqrt{2}}{-7}\right| = \frac{4\sqrt{2}}{7}$$

$\therefore \theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$

The correct option is (B)

137. Let $f(x) = ax^3 + bx^2 + cx, x \in [0, 1]$.

$\therefore f'(x) = 3ax^2 + 2bx + c$.

Since $f(x)$ is a polynomial function of x, it is continuous and differentiable for all $x \in [0, 1]$.

Also, $f(0) = 0; f(1) = a + b + c = 0$.

$\therefore f(0) = f(1)$.

Applying Rolle's theorem, $f'(k) = 0$ for atleast one value k, $0 < k < 1$. Hence, k is a root of the equation

$3ax^2 + 2bx + c = 0$, where $0 < k < 1$.

The correct option is (B)

138. Let $f(x) = (x - 3)\log x$

Then, $f(1) = -2\log 1 = 0$ and $f(3) = (3 - 3)\log 3 = 0$.

As, $(x - 3)$ and $\log x$ are continuous and differentiable in [1, 3], therefore $(x - 3)\log x = f(x)$ is also continuous and differentiable in [1, 3]. Hence, by Rolle's theorem, there exists a value of x in (1, 3) such that

$$f'(x) = 0 \Rightarrow \log x + (x - 3)\frac{1}{x} = 0 \Rightarrow x\log x = 3 - x$$

The correct option is (B)

139. Let α, β $(\alpha < \beta)$ be any two real roots of

$f(x) = e^{-x} - \sin x$.

Then, $f(\alpha) = 0 = f(\beta)$

Moreover, $f(x)$ is continuous and differentiable for $x \in [\alpha, \beta]$.

Hence, from Rolle's theorem, there exists at least one x in (α, β) such that

$f'(x) = 0 \Rightarrow -e^{-x} - \cos x = 0$

$\Rightarrow -e^{-x}(1 + e^x\cos x) = 0 \Rightarrow e^x\cos x = -1$

The correct option is (A)

140. $f(x) = \sqrt{x} \Rightarrow f(A) = \sqrt{a}, f(b) = \sqrt{b}$

$g(x) = \frac{1}{\sqrt{x}} \Rightarrow g(A) = \frac{1}{\sqrt{a}}, g(b) = \frac{1}{\sqrt{b}}$

$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(C) = \frac{1}{2\sqrt{c}}$

$g'(x) = \frac{-1}{2x^{3/2}} \Rightarrow g'(C) = \frac{-1}{2c\sqrt{c}}$

$$\therefore \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}}$$

$$\Rightarrow \left(\sqrt{b} - \sqrt{a}\right) \times \frac{\sqrt{a}\cdot\sqrt{b}}{\sqrt{a} - \sqrt{b}} = -c$$

$\Rightarrow -\sqrt{ab} = -c\ c = \sqrt{ab}$.

The correct option is (C)

141. Let $f(x) = \sin x$ and $g(x) = \cos x$

Then, $f'(x) = \cos x$ and $g'(x) = -\sin x$

Clearly, f and g are continuous on $[\alpha, \beta]$. f and g are derivable on (α, β) $g'(x) = -\sin x \neq 0$ for any $x \in (\alpha, \beta)$

$\therefore$ By Cauchy's Mean Value theorem on $[\alpha, \beta]$, we have,

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(\theta)}{g'(\theta)}, \theta < \alpha < \beta$$

$$\Rightarrow \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta}$$

$$\Rightarrow \frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, \alpha < \beta < \beta.$$

The correct option is (B)

142. $f(x) = e^x \Rightarrow f(a) = e^a, f(b) = e^b$

$g(x) = e^{-x} \Rightarrow g(a) = e^{-a}, g(b) = e^{-b}$

$f'(x) = e^x \Rightarrow f'(c) = e^c$

$g'(x) = -e^{-x} \Rightarrow g'(c) = -e^{-c}$

$$\therefore \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}$$

$$\Rightarrow \frac{e^b - e^a}{e^{1/b} - e^{1/a}} = -e^{2c}$$

$$\Rightarrow \frac{(e^b - e^a) \times e^a e^b}{e^a - e^b} = -e^{2c}$$

$\Rightarrow -e^{a+b} = -e^{2c} \Rightarrow a + b = 2c$

$\therefore c = \frac{a + b}{2}$

The correct option is (A)

Match the Column Type

143. **I.** We have,

$f(x) = (1 + b^2)\left[x^2 + \frac{2b}{1+b^2}x + \frac{b^2}{(1+b^2)^2}\right] - \frac{b^2}{1+b^2} + 1$

$= (1 + b^2)\left(x + \frac{b}{1+b^2}\right)^2 + \frac{1}{1+b^2} \geq \frac{1}{1+b^2}$

$\therefore \quad m(b) = \frac{1}{1+b^2}$. So, range of $m(b) = (0, 1]$.

The correct option is (A)

II. Let $f(x) = \log(1 + x) - x$

$\Rightarrow \; f'(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x}$

$\Rightarrow \; f'(x) < 0$, for $x > 0$

$\Rightarrow \; f(x)$ is decreasing for $x > 0$

$\Rightarrow \; f(x) < f(0)$, for $x > 0$

$\Rightarrow \; \log(1 + x) - x < 0$, for $x > 0$

i.e., $\log(1 + x) < x$, for $x > 0$.

The correct option is (C)

III. Let

$f(x) = a_0\frac{x^{n+1}}{n+1} + a_1\frac{x^n}{n} + a_2\frac{x^{n-1}}{n-1} + \ldots + a_{n-1}\frac{x^2}{2} + a_n x$.

Then, $f(x)$ is continuous and differentiable in [0, 1], as it is a polynomial function of x.

Also, $f(0) = 0$

and, $f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots + \frac{a_{n-1}}{2} + a_n = 0.$

Hence, by Rolle's theorem, there exists at least one real number $c \in (0, 1)$ such that $f'(c) = 0$, i.e., c is a root of the equation $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$.

The correct option is (B)

IV. Let $f(x) = ax^4 + bx^3 + cx^2 + dx$

Then, $f(0) = 0$ and $f(3) = 81a + 27b + 9c + 3d$

$= 3(27a + 9b + 3c + d)$

$= 0 \qquad (\because 27a + 9b + 3c + d = 0)$

Therefore, 0 and 3 are roots of the polynomial $f(x)$. So, by Rolle's theorem, there exists at least one root of the polynomial $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$ lying between 0 and 3.

The correct option is (D)

144. **I.** Clearly, $f(x)$ decreasing just before $x = 3$ and increasing after $x = 3$. For $x = 3$ to be a point of local minima,

$f(3) \geq f(3 - 0) \Rightarrow \; -15 \geq 12 - 27 + \ln(a^2 - 3a + 3)$

$\Rightarrow \; 0 < a^2 - 3a + 3 \leq 1$

$\Rightarrow \; 1 \leq a \leq 2.$

The correct option is (C)

II. We have,

$f(x) = \left(\frac{\sqrt{a+1}}{a-1} - 1\right)x^3 - x + \ln(a - 1)$

defined $\forall\, x \in R, a > 1$

and, $f'(x) = 3\left(\frac{\sqrt{a+1}}{a-1} - 1\right)x^2 - 1$

Clearly, $f'(x)$ is negative $\forall\, x \in R$, if

$\frac{\sqrt{a+1}}{a-1} - 1 < 0$

$\Rightarrow \; \sqrt{a+1} < a - 1 \qquad [a > 1]$

$\Rightarrow \; a + 1 < a^2 - 2a + 1$

$\Rightarrow \; a(a - 3) > 0$

$\Rightarrow \; a \in (-\infty, 0) \cup (3, \infty)$

Since $a > 1$, therefore we have $a \in (3, \infty)$.

The correct option is (D)

III. We have,

$f(x) = x^3 + ax^2 + a^2x + 2\sin^2 x$

and, $f'(x) = 3x^2 + 2ax + a^2 + 2\sin 2x$

which is positive $\forall\, x \in R$, if

$3x^2 + 2ax + a^2 - 2 > 0 \;\forall\, x \in R$

which is true, if discriminant < 0

i.e., $a^2 - 3(a^2 - 2) < 0$

i.e., $a^2 > 3$

i.e., $a \in R \sim \left[-\sqrt{3}, \sqrt{3}\right]$.

The correct option is (A)

IV. We have,

$f(x) = |e^{ax} - e^{-ax}|, (a > 0)$

$\Rightarrow \; f(x) = \begin{cases} e^{-ax} - e^{ax}, & x < 0 \\ e^{ax} - e^{-ax}, & x \geq 0 \end{cases}$

$\Rightarrow \; f'(x) = \begin{cases} -a\left(e^{-ax} + e^{ax}\right), & x < 0 \\ a\left(e^{ax} + e^{-ax}\right), & x \geq 0 \end{cases}$

Then, we have for $a > 0$

$f'(x) < 0 \;\pi\, x < 0$

$\Rightarrow \; f$ is decreasing

and, $f'(x) > 0 \;\pi\, x > 0$

$\Rightarrow \; f$ is increasing.

The correct option is (B)

Assertion-Reason Type

145. We have, $y = x^2 + bx + c$

$\Rightarrow \quad \frac{dy}{dx} = 2x + b$

Since the curve touches the line $y = x$ at the point (1, 1)

$\therefore \quad (2x + b)]_{(1,1)} = 1$ i.e., $2 + b = 1 \quad \Rightarrow \quad b = -1$

Also, the curve passes through the point (1, 1)

$\therefore \quad 1 = 1 + b + c$ i.e., $c = -b = 1$

$\therefore \quad y = x^2 - x + 1 \Rightarrow \frac{dy}{dx} = 2x - 1.$

Now, $\frac{dy}{dx} < 0 \Rightarrow \quad 2x - 1 < 0 \quad \Rightarrow \quad \boldsymbol{x} < \frac{1}{2}$

The correct option is (A)

146. We have, $f(x) = \frac{\sin x}{x}$

$\Rightarrow \quad f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x(x - \tan x)}{x^2}$

But $\tan x > x$ and $\cos x > 0$, for $0 < x < \frac{\pi}{2}$

$\therefore \quad f'(x) < 0$ in the interval $\left(0, \frac{\pi}{2}\right)$

Thus, $f(x)$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

The correct option is (A)

147. Let $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

Then, $f'(x) = \frac{x\cos x - \sin x}{x^2}$

$= \frac{\cos x(x - \tan x)}{x^2} < 0$ if $x \in \left(0, \frac{\pi}{2}\right)$.

$\left(\because \tan x > x \text{ and } \cos x > 0 \text{ when } 0 < x < \frac{\pi}{2}\right)$

$\therefore \quad f(x)$ is decreasing in $\left(0, \frac{\pi}{2}\right)$. Since $0 < x < \frac{\pi}{2}$,

$\therefore \quad f\left(\frac{\pi}{2}\right) < f(x) < f(0) \quad \Rightarrow \quad \frac{2}{\pi} < \frac{\sin x}{x} < 1$

The correct option is (C)

148. We know that $\sin x < x$ if $0 < x < \frac{\pi}{2}$ (1)

Since $\cos x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$, $\cos(\sin x) > \cos x$.

Also, since $0 < x < \frac{\pi}{2}$, π $0 < \cos x < 1 < \frac{\pi}{2}$

$\pi \sin(\cos x) < \cos x$ [Using (1)]

Hence $\cos(\sin \boldsymbol{x}) > \cos \boldsymbol{x} > \sin(\cos \boldsymbol{x})$ if $0 < x < \frac{\pi}{2}$

The correct option is (A)

149. Let $f(x) = x \tan x$

$\Rightarrow \quad f'(x) = \tan x \times 1 + x\sec^2 x > 0$, for $x \in \left(0, \frac{\pi}{2}\right)$.

So, $f(x)$ is increasing for $x \in \left(0, \frac{\pi}{2}\right)$.

Since, $0 < \alpha < \beta < \frac{\pi}{2}$. $\pi f(\alpha) > f(\beta)$

$\Rightarrow \quad \alpha \tan \beta > \alpha \tan \beta$ i.e., $\frac{\tan \beta}{\tan \alpha} > \frac{\alpha}{\beta}$

The correct option is (A)

150. Since $g(x)$ is decreasing,

$\therefore \quad g(x_2) \leq g(x_1)$ when $x_2 \geq x_1$.

Since $f(x)$ is increasing,

$\therefore \quad f[g(x_1)] \geq f[g(x_2)]$

$\Rightarrow \quad h(x_1) \geq h(x_2)$ when $x_2 \geq x_1$.

$\Rightarrow \quad h(x)$ is a decreasing function of x and $h(0) = 0$.

Also, domain of $h = [0, \infty)$ and range of $h = [0, \infty)$.

$\therefore \quad h(x) = 0, \forall x \in [0, \infty)$.

The correct option is (C)

151. We have, $f'(x) = \frac{1}{1 + x^2}$ for all x. (1)

$\Rightarrow \quad f'(x) > 0$, for all x $\quad (\because 1 + x^2 > 0)$

From (1), it follows that $f(x)$ is differentiable at all x, therefore, $f(x)$ is also continuous at all x.

$\therefore$ By mean value theorem in [0, 2]

$\frac{f(2) - f(0)}{2 - 0} = f'(c) = \frac{1}{1 + c^2}$ where $0 < c < 2$

$\Rightarrow \quad \frac{f(2) - 0}{2} = \frac{1}{1 + c^2}$ or $f(2) = \frac{2}{1 + c^2}$ (2)

Now, $0 < c < 2$,

$\therefore \quad \frac{2}{1 + c^2} < \frac{2}{1 + 0^2}$ or $\frac{2}{1 + c^2} < 2$ (3)

and, $\frac{2}{1 + c^2} > \frac{2}{1 + 2^2} = \frac{2}{5} = 0.4$

or, $\frac{2}{1 + c^2} > 0.4$ (4)

From (2), (3) and (4) it follows that $0.4 < f(2) < 2$.

The correct option is (A)

152. We have,

$f(x) = \tan x, x \in [0, \pi/7]$

and, $f'(x) = \sec^2 x, x \in [0, \pi/7]$

Applying Lagrange's theorem on $f(x)$ in the interval $[0, \pi/7]$, we have

$$f'(C) = \frac{f(\pi/7) - f(0)}{(\pi/7) - 0} \text{ for some c in } \left[0, \frac{\pi}{7}\right]$$

Since, $\sec^2 x$ is strictly increasing in $\left[0, \frac{\pi}{7}\right]$, therefore we have,

$f'(0) < f'(c) < f'(\pi/7)$

i.e., $1 < \frac{f(\pi/7)}{(\pi/7)} < \sec^2(\pi/7) < \sec^2(\pi/4) = 2$

i.e., $\frac{\pi}{7} < f(\pi/7) < \frac{2\pi}{7}$

The correct option is (A)

153. We have,

$f(x) = \ln(\ln x), x > 1$

and, $f'(x) = \frac{1}{x\, ln\,(x)}, x > 1$

Since f is continuous and differentiable in $(1, \infty)$, therefore applying Lagrange's theorem on f in (a, b), we have

$$f'(C) = \frac{f(b) - f(a)}{b - a} \text{ for some } c \text{ in } (a, b)$$

Since, $\frac{1}{x\, ln\, x}$ is strictly decreasing in (a, b), therefore we have

$f'(b) < f'(c) < f'(a)$

i.e., $\frac{1}{b\, ln\, b} < \frac{f(b) - f(a)}{b - a} < \frac{1}{a\, ln\, a}$

The correct option is (A)

154. Let $f(x) = \sin(\tan x) - x, x \in [0, \pi/4]$

Then, $f'(x) = \sec^2 x \cos(\tan x) - 1$

$= \tan^2 x \cos(\tan x) + \cos(\tan x) - 1$

$\geq \tan^2 x \cos(\tan x) - \frac{\tan^2 x}{2}$

$\left[\text{using } \cos x - 1 \geq \frac{-x^2}{2}\right]$

$= \tan^2 x \left[\cos(\tan x - \frac{1}{2})\right]$

In the interval $0 \leq x \leq \pi/4$, we have $0 \leq \tan x \leq 1$

$\Rightarrow \cos 1 \leq \cos(\tan x) \leq \cos \theta = 1$

Now, $1 < \pi/3 \Rightarrow \cos 1 > \cos \theta/3 = 1/2$

Thus, $\cos(\tan x) > 1/2\ \pi x \in [0, \pi/4]$

$\Rightarrow f'(x) \geq 0\ \pi x \in [0, \pi/4]$

$\Rightarrow f(x)$ increases in $[0, \pi/4]$

$\Rightarrow f(x) > f(0) = 0.$

The correct option is (A)

155. We have,

$f(x) = (1 + x)^p - x^p - 1, x \geq 0$ and $0 < p \leq 1$

Then, $f(x) = p(1 + x)^{p-1} - px^{p-1}$

$$= p\left[\frac{1}{(1+x)^{1-p}} - \frac{1}{x^{1-p}}\right] < 0\ \forall\ x > 0$$

$\Rightarrow f(x)$ strictly decreases in $(0, \infty)$

$\Rightarrow f(x) \leq f(0) = 0$

i.e., $(1 + x)^p \leq 1 + x^p\ \forall\ x \geq 0$ and $0 < p \leq 1$

Now, putting $x = \frac{a}{b}$ and $p = \frac{1}{n}$ $(n \geq 1)$, we get

$$\left(\frac{a}{b} + 1\right)^{1/n} \leq \left(\frac{a}{b}\right)^{1/n} + 1$$

$\Rightarrow (a + b)^{1/n} \leq a^{1/n} + b^{1/n}$

The correct option is (A)

156. We have,

$f(x) = \frac{\ln x}{x},\quad x > 0$

Then, $f'(x) = \frac{1 - \ln x}{x^2} < 0\ \forall\ x > e$

$\Rightarrow f(x)$ strictly decreases in (e, ∞)

Thus, we have

$f(303) < f(202)$

$\Rightarrow \frac{\ln(303)}{303} < \frac{\ln(202)}{202}$

$\Rightarrow 202 \ln(303) < 303 \ln(202) \Rightarrow 303^{202} < 202^{303}$

The correct option is (C)

157. Let $f(x) = \ln x - x, x > 0$

Then, $f'(x) = \frac{1}{x} - 1 = \frac{1 - x}{x} < 0\ \forall x > 1$

$> 0\ \forall\ 0 < x < 1$

$\Rightarrow f(x)$ strictly increases in $(0, 1)$

and strictly decreases in $(1, \infty)$

$\Rightarrow f(x)$ has greatest value at $x = 1$

$\Rightarrow f(x) \leq f(1) = -1 < 0$

i.e., $\ln x < x$

Now, we have

$e^{-\theta/2} < \theta < \theta/2$

i.e., $0 < \theta < \theta/2$

i.e., $\cos 0 > \cos \theta > \cos \theta/2$

i.e., $0 < \cos \theta < 1$

i.e., $\ln(\cos \theta) < 0$ $[\because \ln x < 0\ \forall x \in (0, 1)]$ (1)

Also, we have

$e^{-\theta/2} < \theta < \theta/2$

i.e., $-\theta/2 < \ln \theta < \ln \theta/2$ [$\because$ $\ln x$ is increasing in $(0, \infty)$]

i.e., $-\theta/2 < \ln \theta < \theta/2$ [Using $\ln x < x$]

i.e., $0 < \cos(\ln \theta) < 1$ (2)

From (1) and (2), we get

$\ln(\cos \theta) < \cos(\ln \theta)$

The correct option is (A)

Previous Year's Questions

158. The equations of two curves are

$x^3 - 3xy^2 + 2 = 0$ (1)

and $3x^2y - y^3 - 2 = 0$ (2)

On differentiating the above Equations (1) and (2) with respect to x, we get

$$\left(\frac{dy}{dx}\right)_{c_1} = \frac{x^2 - y^2}{2xy}$$

and $$\left(\frac{dy}{dx}\right)_{c_2} = \frac{-2xy}{x^2 - y^2}$$

$$\left(\frac{dy}{dx}\right)_{c_1} \times \left(\frac{dy}{dx}\right)_{c_2} = \left(\frac{x^2 - y^2}{2xy}\right)\left(\frac{-2xy}{x^2 - y^2}\right)$$

$$= -1$$

Hence, the two curves cut at right angle.

The correct option is (A)

159. Key Idea :A function $f(x)$ is said to be increasing function, if $f'(x) > 0$.

$\because f(x) = \cot^{-1} x + x$

$\therefore f'(x) = -\dfrac{1}{1+x^2} + 1 = \dfrac{x^2}{1+x^2} > 0$

Hence, the above $f(x)$ is increasing function since $f'(x) > 0$ for all x.

The correct option is (A)

160. We have that

$f(x) = (x+1)^{1/3} - (x-1)^{1/3}$

$$\therefore f'(x) = \frac{1}{3}\left[\frac{1}{(x+1)^{2/3}} - \frac{1}{(x-1)^{2/3}}\right]$$

$$= \frac{(x-1)^{2/3} - (x+1)^{2/3}}{3(x^2-1)^{2/3}}$$

Clearly, the derivative $f'(x)$ does not exist at $x = \pm 1$. Now, $f'(x) = 0$, implies that

$(x-1)^{2/3} = (x+1)^{2/3}$.

This gives $x = 0$

Clearly, $f'(x) \neq 0$ for any other value of $x \in [0,1]$. The value of $f(x)$ at $x = 0$ is 2.

Hence, the greatest value of $f(x)$ is 2

The correct option is (C)

161. $f(x) = 0$ at $x = p, q$

$\therefore 6p^2 + 18ap + 12a^2 = 0$

and, $6q^2 + 18aq + 12a^2 = 0$

$f''(x) < 0$ at $x = p$

and $f''(x) > 0$ at $x = q$.

Clearly $f''(x) > 0$ for $x = 2a => q = 2a < 0$ for $x = a => p = a$

or $p^2 = q \Rightarrow a = 2$.

The correct option is (C)

162. $f''(x) = 6(x-1) \Rightarrow f'(x) = 3(x-1)^2 + c$

and $f'(2) = 3 \Rightarrow c = 0$

Therefore, $f(x) = (x-1)^3 + k$ and $f(2) = 1$

$\Rightarrow k = 0 \Rightarrow f(x) = (x-1)^3$.

The correct option is (B)

163. Eliminating θ, we get the equation $(x-a)^2 + y^2 = a^2$.

Hence normal always pass through $(a, 0)$.

The correct option is (A)

164. We find $\dfrac{dy}{dx} = \tan\theta$ which implies slope of normal $= -\cot\theta$

Equation of normal at θ is

$y - a(\sin\theta - \theta\cos\theta) = -\cot\theta\,(x - a(\cos\theta + \theta\sin\theta))$

$\Rightarrow y\sin\theta - a\sin^2\theta + a\,\theta\cos\theta\sin\theta = -x\cos\theta + a\cos^2\theta + a\,\theta\sin\theta\cos\theta$

$\Rightarrow x\cos\theta + y\sin\theta = a$

Clearly this is an equation of straight line which is at a constant distance a from origin.

The correct option is (D)

165. The function $f(x) = 3x^2 - 2x + 1$ is increasing when $f'(x) = 6x - 2 \geq 0 \Rightarrow x \in [1/3, \infty]$

The correct option is (C)

166. Given that $f(1) = -2$ and $f'(x) \geq 2\ \forall\ x \in [1, 6]$

Applying Lagrange's mean value theorem

$\dfrac{f(6) - f(1)}{5} = f'(c) \geq 2$

$\Rightarrow f(6) \geq 10 + f(1)$

$\Rightarrow f(6) \geq 10 - 2$

$\Rightarrow f(6) \geq 8$.

The correct option is (A)

167. $\dfrac{dv}{dt} = 50$

$\Rightarrow 4\pi r^2 \dfrac{dr}{dt} = 50$

$\Rightarrow \dfrac{dr}{dt} = \dfrac{50}{4\pi(15)^2}$ where r = 15

$= \dfrac{1}{16\pi}$

The correct option is (B)

168. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x$, then $f(0) = 0$.

Also, $f(\alpha) = 0$

By Roll's theorem, $f'(k) = 0$ for some $k \in (0, \alpha)$.

The correct option is (B)

169. $\dfrac{x}{2} + \dfrac{2}{x}$ is of the form $x + \dfrac{1}{x} \geq 2$ and equality holds for $x = 1$

The correct option is (A)

170. Given curve implies that $\dfrac{dy}{dx} = 2x - 5$

$\therefore \quad m_1 = (2x-5)_{(2,0)} = -1,\ m_2 = (2x-5)_{(3,0)} = 1$

$\Rightarrow \quad m_1 m_2 = -1$

The correct option is (B)

171. Equation of the normal is

$Y - y = -\dfrac{dx}{dy}(X - x)$

$\Rightarrow \quad G \equiv \left(x + y\dfrac{dy}{dx}, 0\right)$

So, $\left|x + y\dfrac{dy}{dx}\right| = |2x|$

$\Rightarrow y\dfrac{dy}{dx} = x$ or $y\dfrac{dy}{dx} = -3x$

$\Rightarrow \quad ydy = x\,dx \quad$ or $\quad y\,dy = -3xdx$

$\Rightarrow \quad \dfrac{y^2}{2} = \dfrac{x^2}{2} + c, or \dfrac{y^2}{2} = -\dfrac{3x^2}{2} + c$

$\Rightarrow \quad x^2 - y^2 = -2c \quad$ or $\quad 3x^2 + y^2 = 2c.$

The correct option is (A) and (D)

172. Using Lagrange's mean value theorem, we write

$f'(c) = \dfrac{f(3) - f(1)}{3 - 1}$

$\Rightarrow \dfrac{1}{c} = \dfrac{\log 3 - \log 1}{2}$

$\Rightarrow c = \dfrac{2}{\log_e 3} = 2\log_3 e$

The correct option is (A)

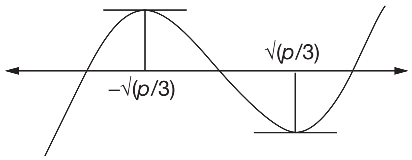
173. Point must be on the directrix of the parabola.

Hence the required point is (–2, 0).

The correct option is (D)

174. Let $f(x) = x^3 - px + q$

Now for the extrema,



$f'(x) = 0$

$\Rightarrow \quad 3x^3 - p = 0$

$\Rightarrow x^2 = \dfrac{p}{3}$

$\therefore x = \pm\sqrt{\dfrac{p}{3}}$

The correct option is (A)

175. $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$

Let $\quad f(x) = x^7 + 14x^5 + 16x^3 + 30x$

$\Rightarrow \quad f'(x) = 7x^6 + 70x^4 + 48x^2 + 30 > 0\ \forall\ x.$

$\therefore \quad f(x)$ is an increasing function $\forall\ x$.

So it intersects the x-axis at exactly one point.

The correct option is (B)

176. Since $P(x) = x^4 + ax^3 + bx^2 + cx + d$,

$P'(x) = 4x^3 + 3ax^2 + 2bx + c$

$\because \quad x = 0$ is a solution for $P'(x) = 0 \quad \Rightarrow \quad c = 0$

$\therefore \quad P(x) = x^4 + ax^3 + bx^2 + d \qquad (1)$

Also, we have $P(-1) < P(1)$

$\Rightarrow \quad 1 - a + b + d < 1 + a + b + d \quad \Rightarrow \quad a > 0$

$\because \quad P'(x) = 0$, only when $x = 0$ and $P(x)$ is differentiable in (–1, 1), we should have the maximum and minimum at the points $x = -1$, 0 and 1 only

Also, we have $P(-1) < P(1)$

$\therefore \quad$ Max. of $P(x)$ = Max. {P(0), P(1)} and Min. of $P(x)$ = Min. {$P(-1)$, $P(0)$}

In the interval [0, 1],

$P'(x) = 4x^3 + 3ax^2 + 2bx = x\,(4x^2 + 3ax + 2b)$

$\because \quad P'(x)$ has only one root $x = 0$, $4x^2 + 3ax + 2b = 0$ has no real roots.

$\therefore \quad (3a)^2 - 32b < 0 \quad \Rightarrow \quad \dfrac{3a^2}{32} < b$

$\therefore \quad b > 0$

Thus, we have $a > 0$ and $b > 0$

$\therefore \quad P'(x) = 4x^3 + 3ax^2 + 2bx > 0,\ \forall\ x \in (0, 1)$

Hence $P(x)$ is increasing in [0, 1]

$\therefore \quad$ Max. of $P(x) = P(1)$

Similarly, $P(x)$ is decreasing in [–1, 0]

Therefore Min. $P(x)$ does not occur at $x = -1$.

The correct option is (B)

177. Given line

$x - y + 1 = 0$ (1)

And the parabola

$x = y^2$

implies $1 = 2y\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2y} =$ Slope of given line (1)

$\dfrac{1}{2y} = 1 \Rightarrow y = \dfrac{1}{2} \Rightarrow y = \dfrac{1}{2} \Rightarrow x = \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$

$\Rightarrow (x, y) = \left(\dfrac{1}{4}, \dfrac{1}{2}\right)$

$\therefore$ The shortest distance is $\dfrac{\left|\frac{1}{4} - \frac{1}{2} + 1\right|}{\sqrt{1+1}} = \dfrac{3}{4\sqrt{2}} = \dfrac{3\sqrt{2}}{8}$

The correct option is (A)

178. Parallel to x-axis

$\Rightarrow \dfrac{dy}{dx} = 0$

$\Rightarrow 1 - \dfrac{8}{x^3} = 0$

$\Rightarrow \quad x = 2$

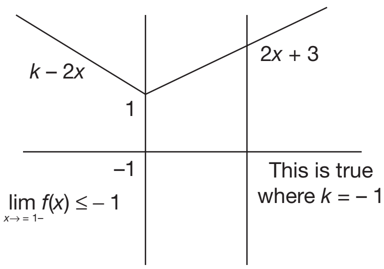
$\Rightarrow \quad y = 3$

Equation of tangent is $y - 3 = 0(x - 2)$

$\Rightarrow \quad y - 3 = 0$

The correct option is (C)

179. $f(x) = \begin{cases} k - 2x, & \text{if } x \le -1 \\ 2x + 3 & \text{if } x > -1 \end{cases}$



The correct option is (C)

180. $v = \dfrac{4}{3}\pi r^2$

After 49 minutes volume = $4500\,\pi - 49\,(72\,\pi) = 972\,\pi$

$\dfrac{4}{3}\pi r^3 = 972\pi$

$\Rightarrow r^3 = 729$

$\Rightarrow r = 9$

$v = \dfrac{4}{3}\pi r3$

$\dfrac{dv}{dt} = \dfrac{4}{3}\pi 3r^2 \dfrac{dr}{dt}$

$72\pi = 4\pi r^2 \dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{72}{4 \cdot 9 \cdot 9} = \dfrac{2}{9}$

The correct option is (C)

181. $f'(x) = \dfrac{1}{x} + 2bx + a$

f has extremevalues and differentiable

$\Rightarrow f'(-1) = 0$

$\Rightarrow a - 2b = 1$

$f'(2) = 0$

$\Rightarrow a + 4b = -\dfrac{1}{2}$

$\Rightarrow a = \dfrac{1}{2}, b = -\dfrac{1}{4}$

$f''(-1), f''(2)$ are negative. f has local maxima at –1, 2

The correct option is (B)

182. Let $h(f) = f(x) - 2g(x)$

as $h(0) = h(1) = 2$

Hence, using Rolle's theorem

$h'(c) = 0$

$\Rightarrow f'(c) = 2g'(c)$

The correct option is (D)

183. $f'(x) = \dfrac{\alpha}{x} + 2\beta x + 1$

$2\beta x^2 + x + \alpha = 0$ has roots –1 and 2

The correct option is (C)

184. $x^2 + 3xy - xy - 3y^2 = 0$

$\Rightarrow \; x(x + 3y) - y(x + 3y) = 0$

$\Rightarrow \; (x + cy)(x - y) = 0$

Equation of normal is $(y - 1) = -(x - 1)$

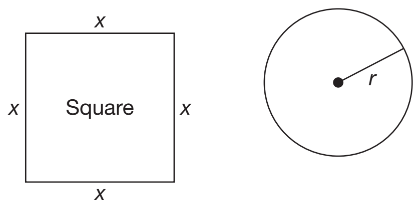
$\Rightarrow x + y = 2$

It intersects $x + 3y = 0$ at the point $(3, -1)$

And hence meets the curve again in the 4^{th} quadrant.

The correct option is (C)

185.



According to the question,

$4x + 2\pi r = 2$

$\Rightarrow 2x + 4\pi r = 1$

$\therefore r = \dfrac{1 - 2x}{\pi}$ (1)

Now. Area $A = x^2 + \pi r^2$

$$= x^2 + \frac{1}{\pi}(2x-1)^2$$

For area A to be minimum, we have

$$\frac{dA}{dx} = 0 \text{ gives } x = \frac{2}{\pi+4} \quad (2)$$

From (1) and (2), we get

$$r = \frac{1}{\pi+4} \quad (3)$$

$\therefore x = 2r$

The correct option is (D)

186. $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$ where $x \in \left(0, \frac{\pi}{2}\right)$

$$= \tan^{-1}\left(\sqrt{\frac{(1+\sin x^2)}{1-\sin^2 x}}\right)$$

$$= \tan^{-1}\left(\frac{1+\sin x}{|\cos x|}\right)$$

$$= \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right) \left(\text{as } x \in \left(0, \frac{\pi}{2}\right)\right)$$

$$= \tan^{-1}\left(\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$$

$$f(x) = \frac{\pi}{4} + \frac{x}{2} \text{ as } x \in \left(0, \frac{\pi}{2}\right) \Rightarrow f'\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$\therefore$ Equation of normal

$$\left(y - \frac{\pi}{3}\right) = -2\left(x - \frac{\pi}{6}\right)$$

The correct option is (C)