CHAPTER 22

Three Dimensional Geometry

Chapter Highlights

ORIGIN

Let $X'OX$, $Y'OY$, $Z'OZ$ be three mutually perpendicular lines which intersect at O. Then O is called the *origin*.

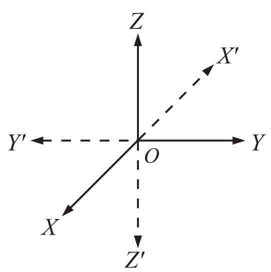


Fig. 22.1

COORDINATE AXES

$X'OX$ is called the x-axis, $Y'OY$ the y-*axis*, $Z'OZ$ the z-axis and taken together these are called the *coordinate axes.*

Coordinate Planes

1. XOY is called the xy-plane.
2. YOZ is called the yz-plane.
3. ZOX is called the zx-plane.

These three, taken together, are called the *coordinate planes*.

The three co-ordinate planes divide the space into eight parts and these parts are called *octants.*

Coordinates

Let P be any point in space. Draw PL, PM, PN perpendiculars to the xy, yz and xz planes, then

1. LP is called the x-coordinate of P.
2. MP is called the y-coordinate of P.
3. NP is called the z-coordinate of P.

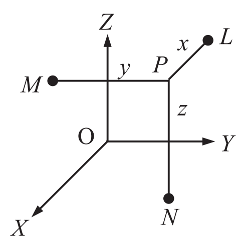


Fig. 22.2

These three, taken together, are known as coordinates of P.

Sign of co-ordinates of a point

The signs of the co-ordinates of a point in three dimension follow the convention that all distances measured along or parallel to OX, OY, OZ will be positive and distances moved along or parallel to OX', OY', OZ' will be negative.

TRICK(S) FOR PROBLEM SOLVING

- Coordinates of any point on x-axis are of the form $(a, 0, 0)$. Similarly, the coordinates of any point on the y-axis are of the form $(0, b, 0)$, while those of points on z-axis are of the form $(0, 0, c)$.
- Coordinates of any point in the xy plane are of the form $(a, b, 0)$. Similarly, the coordinates of any point in the yz plane are of the form $(0, b, c)$, while those of any point in the zx plane are of the form $(a, 0, c)$.
- The equation of xy plane is $z = 0$. Similarly, the equations of yz plane and zx plane are $x = 0$ and $y = 0$ respectively.

DISTANCE FORMULA

1. **Distance formula:** The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

2. **Distance from origin:** Let O be the origin and $P(x, y, z)$ be any point, then

$$OP = \sqrt{(x^2 + y^2 + z^2)}$$

IMPORTANT POINTS

- For any point $P(x, y, z)$, the position vector $\vec{r}$ of P is given by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Conversely, the coordinates of any point whose position vector is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ are $P(x, y, z)$.
- The distance of any point $P(x, y, z)$ having position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ from the origin is $\sqrt{x^2 + y^2 + z^2}$ which is equivalent to the modulus of the vector $\overrightarrow{PQ}$.

DISTANCE OF A POINT FROM COODINATE AXES

Let $P(x, y, z,)$ by any point in the space. Let PA, PB and PC be the perpendiculars drawn from P to the axes OX. OY and OZ respectively.

Then,
$$PA = \sqrt{(y^2 + z^2)}$$
$$PB = \sqrt{(z^2 + x^2)}$$
$$PC = \sqrt{(x^2 + y^2)}$$

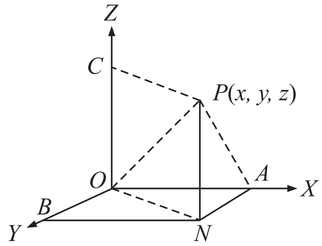


Fig. 22.3

Section Formula

1. The coordinates of the point dividing the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m_1 : m_2$ internally are

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$$

2. The coordinates of the point dividing the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m_1 : m_2$ externally are

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \frac{m_1z_2 - m_2z_1}{m_1 - m_2}\right)$$

3. The coordinates of the mid-point of the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

VECTOR FORM

1. **Internal Division:** If the point R divides the join of PQ internally in the ratio of $m : n$, then position vector of $R(\vec{r})$ is

$$\vec{r} = \frac{m\vec{r_2} + n\vec{r_1}}{m + n}$$

2. **External Division:** If the point R divides the join of PQ externally in the ratio of $m : n$ i.e., internally in the ratio $m : (-n)$, then the position vector $R(\vec{r})$ is

$$\vec{r} = \frac{m\vec{r_2} - n\vec{r_1}}{m - n}$$

3. If is the point R is the mid point of the line joining PQ, then $m : n = 1 : 1$, therefore the position vector $R(\vec{r})$ is

$$\vec{r} = \frac{\vec{r_1} + \vec{r_2}}{2}$$

Coordinates of the general point

The co-ordinates of any point lying on the line joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ may be taken as

$\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}, \frac{kz_2+z_1}{k+1}\right)$, which divides PQ in the ratio $k : 1$. This is called *general point* on the line PQ.

Centroid of a triangle

The coordinates of the centroid of the triangle ABC, whose vertices are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Centroid of a Tetrahedron

If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) are the vertices of a tetrahedron, then its centroid G is given by

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$

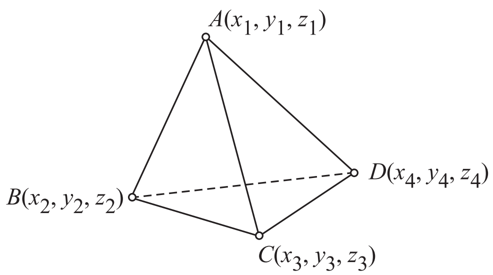


Fig. 22.4

SOLVED EXAMPLES

1. If $P(\vec{p}), Q(\vec{q}), R(\vec{r})$ and $S(\vec{s})$ be four points such that $3\vec{p}+8\vec{q}=6\vec{r}+5\vec{s}$, then the lines PQ and RS and

(A) skew (B) parallel
(C) intersecting (D) none of these

Solution: (D)

Given: $3\vec{p}+8\vec{q}=6\vec{r}+5\vec{s}$

$$\Rightarrow \frac{3\vec{p}+8\vec{q}}{8+3}=\frac{6\vec{r}+5\vec{s}}{5+6}$$

$\Rightarrow$ The point which divides PQ in the ratio $8 : 3$ is the same as the point which divides RS in the ratio $5 : 6$. Hence, the lines PQ and RS intersect.

2. The edge of a cube is of length 'a' then the shortest distance between the diagonal of a cube and an edge skew to it is

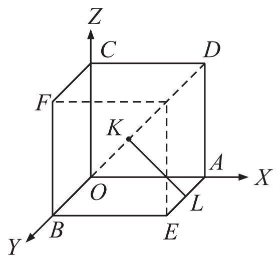
(A) $a\sqrt{2}$ (B) a
(C) $\sqrt{2}/a$ (D) $a/\sqrt{2}$

Solution: (D)

Requaried distance = KL

$$= \sqrt{\left(a-\frac{a}{2}\right)^2+0^2+\left(0-\frac{a}{2}\right)^2}$$

$$= \frac{a}{\sqrt{2}}.$$



DIRECTION COSINES

If a line makes angles α, β, γ with the positive directions of x-axis, y-axis and z-axis respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called its *direction cosines*.

The direction cosines are generally denoted as l, m, n i.e., $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

The angles α, β, γ are known as *direction angles*.

Direction Ratios

Three numbers a, b, c proportional to the direction cosines l, m, n of a line are known as the *direction ratios* of the line.

Thus a, b, c are the direction ratios of a line, provided

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$$

Useful Results on Direction Cosines and Direction Ratios

If $P(x, y, z)$ is a point in space such that $\mathbf{r} = OP$ has direction cosines l, m, n, then

(a) $x = l|\mathbf{r}|, y = m|\mathbf{r}|, z = n|\mathbf{r}|$

(b) $l|\mathbf{r}|, m|\mathbf{r}|, n|\mathbf{r}|$ are projections of $\mathbf{r}$ on OX, OY, OZ respectively.

(c) $\mathbf{r} = |\mathbf{r}|(l\hat{i}+m\hat{j}+n\hat{k})$ and $\hat{r} = l\hat{i}+m\hat{j}+n\hat{k}$

(d) $l^2+m^2+n^2=1$.

(e) If $\mathbf{r} = a\hat{j}+b\hat{j}+c\hat{k}$, then

(a) a, b, c are the direction ratios of $\mathbf{r}$.

(b) Direction cosines of $\mathbf{r}$ are given by

$$l=\frac{a}{\sqrt{a^2+b^2+c^2}}, m=\frac{b}{\sqrt{a^2+b^2+c^2}},$$

$$n=\frac{c}{\sqrt{a^2+b^2+c^2}}$$

(f) Direction ratios of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are x_2-x_1, y_2-y_1, z_2-z_1, and its direction cosines are

$$\frac{x_2 - x_1}{|\mathbf{PQ}|}, \frac{y_2 - y_1}{|\mathbf{PQ}|}, \frac{z_2 - z_1}{|\mathbf{PQ}|}$$

(g) The direction cosines of
(a) $\vec{O}X$ are (1, 0, 0)
(b) $\vec{O}Y$ are (0, 1, 0)
(c) $\vec{O}Z$ are (0, 0, 1)

CAUTION

Direction cosines of a line are unique but direction ratios of a line are not unique and can be infinite.

ANGLE BETWEEN TWO LINES

If θ is the angle between two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2, then

$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$ and

$$\sin\theta = \sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}$$

TRICK(S) FOR PROBLEM SOLVING

- If $l_1l_2 + m_1m_2 + n_1n_2 = 0$, then two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ having direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 are orthogonal.
- If $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ then two vectors are parallel.
- Any vector equally inclined to all the three axes have direction cosines as
$$\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$$
- If any line makes angles $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube, then
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$
- If l_1, m_1, n_1 and l_2, m_2, n_2 are the *d.c.*'s of two concurrent lines, then the *d.c.*'s of the lines bisecting the angles between them are proportional to $l_1 \pm l_2, m_1 \pm m_2, n_1 \pm n_2$.
- The angle between any two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.
- The angle between a diagonal of a cube and the diagonal of a face of the cube is $\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$.
- If the edges of a rectangular parallelopiped be a, b, c, then the angles between the two diagonals are $\cos^{-1}\left[\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right]$.

Angle in terms of Direction Ratios

If θ is the angle between two line having direction ratios a_1, b_1, c_1 and a_2, b_2, c_2, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

TRICK(S) FOR PROBLEM SOLVING

The two line are orthogonal if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

and parallel if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

SOLVED EXAMPLES

3. A vector **r** is inclined at equal angles to *OX*, *OY* and *OZ*. If the magnitude of **r** is 6 units, then **r** is equal to

(A) $\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$ (B) $-\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$

(C) $2\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$ (D) $-2\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$

Solution: (C, D)

Let **r** be inclined at an angle α to each axis, then $l = m = n = \cos\alpha$

Since $l^2 + m^2 + n^2 = 1 \Rightarrow 3\cos^2\alpha = 1 \Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$

If α is acute : $l = m = n = \frac{1}{\sqrt{3}}$:$|\mathbf{r}| = 6$

$$\mathbf{r} = |\mathbf{r}|(l\hat{i} + m\hat{j} + n\hat{k})$$
$$= 6\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right) = 2\sqrt{3}(\hat{i} + \hat{j} + \hat{k}).$$

If α is obtuse $l = m = n = -\frac{1}{\sqrt{3}}$:$|\mathbf{r}| = 6$

$$\mathbf{r} = |\mathbf{r}|(l\hat{i} + m\hat{j} + n\hat{k})$$
$$= 6\left(-\frac{1}{\sqrt{3}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}\right)$$
$$= -2\sqrt{3}(\hat{i} + \hat{j} + \hat{k}).$$

4. If the projection of a line segment on *x*, *y* and *z* axes are respectively 3, 4 and 5, then the length of the line segment is

(A) $3\sqrt{2}$ (B) $5\sqrt{2}$

(C) $5\sqrt{2}$ (D) none of these

Solution: (D)

Let *l*, *m*, *n* be the d.c's of the given line segment *PQ*.

$\therefore$ $l = \cos\alpha, m = \cos\beta, n = \cos\gamma$

where α, β, γ are the angles which the line segment *PQ* makes with the axes.

Suppose length of line segment $PQ = r$.
Thus, projection of line segment PQ on x-axis

$$= PQ\cos\alpha = rl.$$

Also the projection of line segment PQ on x-axis

$$= 3 \text{ (given)}$$

$\therefore \quad lr = 3$

Similarly, $mr = 4$, $nr = 5$.
Now squaring and adding these equations, we get

$$(lr)^2 + (mr)^2 + (nr)^2 = 3^2 + 4^2 + 5^3$$

$\Rightarrow r^2(l^2 + m^2 + n^2) = 9 + 16 + 25$

$\Rightarrow r^2 = 50 \qquad (\because l^2 + m^2 + n^2 = 1)$

$\Rightarrow \quad r = \sqrt{50} = 5\sqrt{2}$

5. A vector **r** is equal inclined with the coordinate axes. If the tip of **r** is in the positive octant and $|\mathbf{r}| = 6$, then **r** is

(A) $2\sqrt{3}(\hat{\imath} - \hat{\jmath} + \hat{k})$ (B) $2\sqrt{3}(-\hat{\imath} + \hat{\jmath} + \hat{k})$
(C) $2\sqrt{3}(\hat{\imath} + \hat{\jmath} - \hat{k})$ (D) $2\sqrt{3}(\hat{\imath} + \hat{\jmath} + \hat{k})$

Solution: (D)

Let l, m, n be the DC's of **r**. Then $l = m = n$ (given).

$$\therefore \quad l^2 + m^2 + n^2 = 1 \Rightarrow 3l^2 = 1 \Rightarrow l = \frac{1}{\sqrt{3}} = m = n$$

$$\therefore \quad \mathbf{r} = |\mathbf{r}|(l\hat{i} + m\hat{j} + n\hat{k}) = 6\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)$$

$$= 2\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$$

6. Equation of the line passing through (1, 1, 1) and parallel to the plane $2x + 3y + z + 5 = 0$ is

(A) $\dfrac{x-1}{1} = \dfrac{y-1}{2} = \dfrac{z-1}{1}$
(B) $\dfrac{x-1}{-1} = \dfrac{y-1}{1} = \dfrac{z-1}{-1}$
(C) $\dfrac{x-1}{3} = \dfrac{y-1}{2} = \dfrac{z-1}{1}$
(D) $\dfrac{x-1}{2} = \dfrac{y-1}{3} = \dfrac{z-1}{1}$

Solution: (B)

If the direction ratios of the line are l, m, n then it is perpendicular to the normal to the plane.

$\therefore \quad 2l + 3m + n = 0$

And the only values of l, m, n that satisfy this equation are –1, 1, –1.
$\therefore$ (B) is the correct answer.

7. The locus of $x^2 + y^2 + z^2 = 0$ is

(A) a circle (B) a sphere
(C) (0, 0, 0) (D) none of these

Solution: (C)

$x^2 + y^2 + z^2 = 0 \quad \Rightarrow \quad x = 0, y = 0, z = 0.$

8. The locus of a point, such that the sum of the squares of its distances from the planes $x + y + z = 0$, $x - z = 0$ and $x - 2y + z = 0$ is 9, is

(A) $x^2 + y^2 + z^2 = 3$ (B) $x^2 + y^2 + z^2 = 6$
(C) $x^2 + y^2 + z^2 = 9$ (D) $x^2 + y^2 + z^2 = 12$

Solution: (C)

Let the variable point be (α, β, γ), then according to the question

$$\left(\frac{|\alpha + \beta + r|}{\sqrt{3}}\right)^2 + \left(\frac{|\alpha - \gamma|}{\sqrt{2}}\right)^2 + \left(\frac{|\alpha - 2\beta + \gamma|}{\sqrt{6}}\right)^2 = 9$$

$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 = 9.$

So, the locus of the point is $x^2 + y^2 + z^2 = 9$.

9. Perpendicular distance of the point (3, 4, 5) from the y-axis, is

(A) $\sqrt{34}$ (B) $\sqrt{41}$
(C) 4 (D) 5

Solution: (A)

Distance of (α, β, γ) from y-axis is given by

$$d = \sqrt{\alpha^2 + \gamma^2}$$

$\therefore$ Distance (d) of (3, 4, 5) from y-axis is

$$d = \sqrt{3^2 + 5^2} = \sqrt{9 + 25} = \sqrt{34}$$

10. The number of straight lines that are equally inclined to three dimensional coordinate axes, is

(A) 2 (B) 4
(C) 6 (D) 8

Solution: (B)

If α, β, γ are the angles made by the line with x, y, and z-axis respectively, then

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Given $\alpha = \beta = \gamma$, $\quad \therefore \ 3\cos^2\alpha = 1$

or $\qquad \cos\alpha = \pm\dfrac{1}{\sqrt{3}}$

Possible direction cosines are $\left(\pm\dfrac{1}{\sqrt{3}}, \pm\dfrac{1}{\sqrt{3}}, \pm\dfrac{1}{\sqrt{3}}\right)$.

Different sets of Dc's are $\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$,

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

and $\quad \left(\dfrac{-1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$

Thus, four lines are equally inclined to axes.

11. The direction ratios of the line $x - y + z - 5 = 0 = x - 3y - 6$ are

(A) 3, 1, –2 (B) 2, –4, 1

(C) $\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}$ (D) $\frac{2}{\sqrt{41}}, \frac{-4}{\sqrt{41}}, \frac{1}{\sqrt{41}}$

Solution: (A)

If l, m, n are the d.c's of the line, then

$$1 \cdot l - 1 \cdot m + 1 \cdot n = 0$$

and

$$1 \cdot l - 3 \cdot m + 0 \cdot n = 0$$

$$\therefore \quad \frac{l}{0+3} = \frac{m}{1-0} = \frac{n}{-3+1}$$

Hence the d.r's of the line are 3, 1, –2.

PROJECTION

Projection of a line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line whose direction cosines are l, m and n : Let PQ be a line segment where $P \equiv (x_1, y_1, z_1)$ and $Q \equiv (x_2, y_2, z_2)$ and AB be a given line with *d.c.*'s as l, m, n. If the line segment PQ makes angle θ with the line AB, then

Projection of PQ is $P'Q' = PQ \cos\theta$

$$= (x_2 - x_1)\cos\alpha + (y_2 - y_1)\cos\beta + (z_2 - z_1)\cos\gamma$$
$$= (x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$$

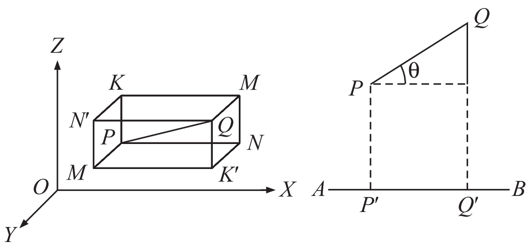


Fig. 22.5

TRICK(S) FOR PROBLEM SOLVING

For x-axis, $l = 1$, $m = 0$, $n = 0$

Hence, projection of PQ on x – axis = $x_2 - x_1$.

Similarly, projection of PQ on y – axis = $y_2 - y_1$ and projection of PQ on z-axis = $z_2 - z_1$.

VECTOR FORM

1. To get the projection of vector $\vec{a}$ along the direction of $\vec{b}$ then take the dot product of $\vec{a}$ with the unit vector along $\vec{b}$

   $\therefore$ Projection of $\vec{a}$ on $\vec{b} = \vec{a}.\hat{b}$

2. If $\vec{r} = a\hat{j} + b\hat{j} + c\hat{k}$ is any vector, then the projection of $\vec{r}$ on a line whose direction cosines are (l, m, n) is

   $$|\vec{r}|\cos q - \vec{r}.(l\hat{i} + m\hat{j} + n\hat{k}) = al + bm + cn$$

   where, $l\hat{i} + m\hat{j} + n\hat{k}$ is the unique unit vector along the line whose direction cosines are given.

STRAIGHT LINE

The vector equation of a straight line passing through a given point with position vector **a** and parallel to a given vector **b** is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

where λ is a scalar.

TRICK(S) FOR PROBLEM SOLVING

- The position vector of any point on the line is taken as $\mathbf{a} + \lambda\mathbf{b}$.
- $\mathbf{r}$ is the position vector of any point $P(x, y, z)$ on the line. Therefore, $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Cartesian Form The equation of a straight line with direction ratios a, b, c and passing through a fixed point (x_1, y_1, z_1) is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

NOTE

- The equation of a line whose direction cosines are l, m, n and which passes through the point (x_1, y_1, z_1) is

  $$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{c}$$

- The coordinates of any point on the line $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ are given by $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, where λ is a real number.
- Equation of x-axis:

  $$\frac{x - 0}{1} = \frac{y - 0}{0} = \frac{z - 0}{0} \text{ or } y = 0, z = 0$$

  Equation of y-axis:

  $$\frac{x - 0}{0} = \frac{y - 0}{1} = \frac{z - 0}{0} \text{ or } x = 0, z = 0$$

  Equation of z-axis:

  $$\frac{x - 0}{0} = \frac{y - 0}{0} = \frac{z - 0}{1} \text{ or } x = 0, y = 0.$$

Vector Equation of a line Passing through two Points

The vector equation of a line passing through two points with position vectors **a** and **b** is

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$$

Cartesian Form The equation of a line passing through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Changing unsymmetrical form to symmetrical form

The unsymmetrical form of a line $ax + by + cz + d = 0$, $a'x + b'y + c'z + d' = 0$ can be changed to symmetrical form as follows:

$$\frac{x-\dfrac{bd'-b'd}{ab'-a'b}}{bc'-b'c}=\frac{y-\dfrac{da'-d'a}{ab'-a'b}}{ca'-c'a}=\frac{z}{ab'-a'b}$$

ANGLE BETWEEN TWO LINES

Vector Form The angle between the two lines

$$\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1 \text{ and } \mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$$

is given by $\cos\theta = \dfrac{\mathbf{b}_1 \cdot \mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}$

Cartesian Form The angle between the two lines

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$$

and $$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$

is given by $$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

INTERSECTION OF TWO LINES

Let the two lines be $\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$ (1)

and $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$ (2)

Step I: Write the coordinates of general points on (1) and (2). The coordinates of general points on (1) and (2) are given by

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}=\lambda$$

and $$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{z_2}=\mu$$

respectively. i.e., $(a_1\lambda + x_1, b_1\lambda + y_1 + c_1\lambda + z_1)$ and $(a_2\mu + x_2, b_2\mu + y_2, c_2\mu + z_2)$.

Step II: If the lines (1) and (2) intersect, then they have a common point.

$$a_1\lambda + x_1 = a_2\mu + x_2, b_1\lambda + y_1 = b_2\mu + y_2$$

and $$c_1\lambda + z_1 = c_2\mu + z_2$$

Step III: Solve any two of the equations in λ and μ obtained in step II. If the values of λ and μ satisfy the third equation, then the lines (i) and (ii) intersect, otherwise they do not intersect.

Step IV: To obtain the coordinates of the point of intersection, substitute the value of λ (or μ) in the coordinates of general point (s) obtained in step 1.

PERPENDICULAR FROM A POINT TO A LINE

Let the equation of the line be

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=r \text{ (say)}$$

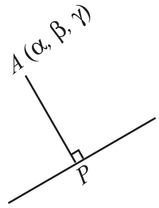


Fig. 22.6

and $A(\alpha, \beta, \gamma)$ be the given point. Then,

1. the coordinates of the foot of the perpendicular from A on the given line are
$$P(lr + a, mr + b, nr + c)$$
2. length of perpendicular (AP) is
$$\sqrt{(lr+a-\alpha)^2+(mr+b-\beta)^2+(nr+c-\gamma)^2}$$
3. equation of the perpendicular is given by
$$\frac{x-\alpha}{lr+a-\alpha}=\frac{y-\beta}{mr+b-\beta}=\frac{z-\gamma}{nr+c-\gamma}$$
where $r = (\alpha - a)l + (\beta - b)m + (\gamma - c)n$

Vector Form

Length of the perpendicular from a point $A(\mathbf{r}_1)$ upon the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ is given by

$$=\frac{|(\mathbf{a}-\mathbf{r}_1)\times\mathbf{b}|}{|\mathbf{b}|}$$

Alternate Method

Find the foot of the perpendicular from the point (1, 6, 3) to line.

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$$

Also, find the length of the perpendicular and the equation of the perpendicular.

Any point on the line $\dfrac{x}{1}=\dfrac{y-1}{2}=\dfrac{z-2}{3}$ can be taken as $(\lambda, 1 + 2\lambda, 2 + 3\lambda)$.

Let this point be P, the foot of perpendicular from A(1, 6, 3) to the line is

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$$

Direction ratios of the given line are 1, 2, 3. Direction ratios of AP are

$$\lambda - 1,\ 1 + 2\lambda - 6,\ 2 + 3\lambda - 3$$

i.e., $$\lambda - 1,\ 2\lambda - 5,\ 3\lambda - 1$$

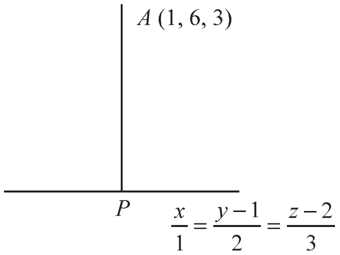


Fig. 22.7

Since, AD is perpendicular to the given line

$\therefore \quad 1(\lambda - 1) + 2(2\lambda - 5) + 3(3\lambda - 1) = 0$

$\Rightarrow \quad \lambda - 1 + 4\lambda - 10 + 9\lambda - 3 = 0$

$\Rightarrow \quad 14\lambda - 14 = 0 \quad \Rightarrow \quad \lambda = 1$

Thus, coordinates of P are (1, 1 + 2, 2 + 3), i.e., (1, 3, 5)

$\therefore$ Foot of perpendicular is (1, 3, 5).

Length of perpendicular is

$$AP = \sqrt{(1-1)^2 + (3-6)^2 + (5-3)^2}$$

$$= \sqrt{0+9+4} = \sqrt{13}$$

Equations of perpendicular, i.e., equations of AP are

$$\frac{x-1}{1-1} = \frac{y-6}{3-6} = \frac{z-3}{5-3}$$

i.e., $$\frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2}$$

SKEW LINES

Two straight lines which are not parallel and which do not intersect, are known as skew lines. Clearly, two skew lines are never coplanar.

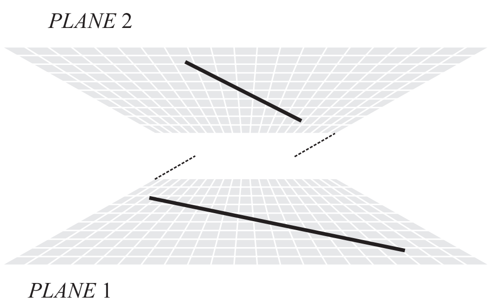


Fig. 22.8

Line of Shortest Distance

If l_1 and l_2 are two skew lines, then the straight line which is perpendicular to each of these two non-intersecting lines is called the "*LIne of shortest distance.*"

IMPORTANT POINTS

There is one and only one line perpendicular to each of lines l_1 and l_2.

Shortest Distance Between two Skew Lines

Let the two skew lines be $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$

Therefore, the shortest distance between the lines is given by

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - l_1n_2)^2 + (l_1m_2 - m_1l_2)^2}}.$$

VECTOR FORM

Shortest distance between the lines

$$\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}_1 \text{ and } \mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}_2 \text{ is}$$

$$d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1)\cdot(\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$

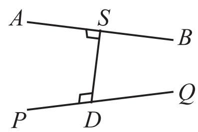


Fig. 22.9

TRICK(S) FOR PROBLEM SOLVING

- Shortest distance between two parallel lines $\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}$ and $\mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}$ is

$$d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}|}{|\mathbf{b}|}$$

- Two lines $\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}_2$ will intersect provided $d = 0$, i.e., when

$$(\mathbf{a}_2 - \mathbf{a}_1)\cdot(\mathbf{b}_1 \times \mathbf{b}_2) = 0$$

The Plane

Plane It is a surface such that if any two points on it are taken, then every point of the line joining them lies on it.

General Equation of a Plane The general equation of a plane is

$$ax + by + cz + d = 0$$

where a, b, c, are not all zero.

IMPORTANT POINTS

- a, b, c are the direction ratios of the normal to the plane $ax + by + cz + d = 0$.
- Equation of yz-plane is $x = 0$
- Equation of zx-plane is $y = 0$
- Equation of xy-plane is $z = 0$
- Equation of any-plane parallel to xy-plane is $z = c$. Similarly, planes parallel to yz-plane and zx-plane are $x = c$ and $y = c$ respectively.

Equation of a Plane in Normal Form

Vector Form If $\hat{n}$ be a unit vector normal to a given plane and p be the length of perpendicular from the origin to the plane, then the equation of the plane is given by,

$$\mathbf{r}\cdot \hat{n} = p$$

Cartesian Form If l, m, n be the direction cosines of the normal to a given plane and p be the length of perpendicular from origin to the plane, then the equation of the plane is $lx + my + nz = p$.

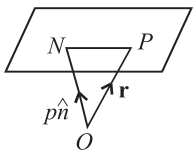


Fig. 22.10

Transformation of General Form to Normal Form To reduce the general equation $ax + by + cz + d = 0$ to normal form, we follow the following working rule:

(a) Write the terms containing x, y and z on left hand side and the constant term on the right hand side.

(b) If the constant term on the right hand side is not positive, make it positive by multiplying both sides by –1.

(c) Divide each term by $\sqrt{a^2+b^2+c^2}$, we get

$$\frac{ax}{\pm\sqrt{a^2+b^2+c^2}} + \frac{by}{\pm\sqrt{a^2+b^2+c^2}} + \frac{cz}{\pm\sqrt{a^2+b^2+c^2}} = \frac{d}{\pm\sqrt{a^2+b^2+c^2}}$$

where + sign is to be taken if $d > 0$ and – sign is to be taken if $d < 0$.

Equation of a Plane Passing through a Given Point

Vector Form The vector equation of a plane through a given point $\mathbf{r}_1$ and perpendicular to $\hat{n}$ is

$$(\mathbf{r}-\mathbf{r}_1)\cdot\hat{n} = 0$$

Cartesian Form The equation of a plane passing through a given point $A(x_1, y_1, z_1)$ and normal to the plane having direction ratios a, b, c is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Equation of a Plane in Intercept Form

If a plane makes intercepts of lengths a, b, c with x-axis, y-axis and z-axis respectively, then the equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Equation of a Plane Passing through Three Points

Vector Form The equation of a plane passing through three points having position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ is $[\,\mathbf{r}-\mathbf{r}_1\ \mathbf{r}-\mathbf{r}_2\ \mathbf{r}-\mathbf{r}_3] = 0$

Cartesian Form The equation of a plane passing through three given points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ x-x_3 & y-y_3 & z-z_3 \end{vmatrix} = 0$$

Equation of a Plane through Two Given Points and Parallel to a Given Vector

Vector Form The equation of a plane through two given points having position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ and parallel to a given vector $\mathbf{m}$ is

$$(\mathbf{r}-\mathbf{r}_1)\cdot[\,(\mathbf{r}_2-\mathbf{r}_1)\times\mathbf{m}\,] = 0$$

Cartesian Form The equation of a plane passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) and parallel to a line having direction ratios a, b, c is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ a & b & c \end{vmatrix} = 0$$

Equation of a Plane Passing through a Given Point and Parallel to Two Given Vectors

Vectors Form The equation of a plane passing through a point having position vector $\mathbf{a}$ and parallel to two given vectors $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}, \text{ where } \lambda \text{ and } \mu \text{ are scalars}$$

or $$(\mathbf{r}-\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c}) = 0 \text{ or } \mathbf{r}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$$

Cartesian Form The equation of a plane passing through a point (x_1, y_1, z_1) and parallel to two lines having direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Planes Parallel to a Given Plane

Cartesian Form Equation of a plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$, where k is a constant to be detemined by the given condition.

Vector Form The equation of a plane parallel to the plane $\mathbf{r} \,.\, \mathbf{n} = d_1$ is $\mathbf{r} \,.\, \mathbf{n} = d_2$, where d_2 is a constant to be determined by the given condition.

Angle between Two Planes

Angle between two planes is the angle between their normals.

Vector Form The angle between the planes $\mathbf{r} \,.\, \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = d_2$ is given by,

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_1}{|\mathbf{n}_1||\mathbf{n}_2|}$$

Cartesian Form The angle between the planes

$a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

TRICK(S) FOR PROBLEM SOLVING

- If $a_1a_2 + b_1b_2 + c_1c_2 = 0$, then the planes are perpendicular to each other.
- If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the planes are parallel to each other.

Angle between a Line and a Plane

The angle between a line and a plane is the angle between the line and the normal to the plane.

Vector Form If θ is the angle between the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and the plane $\mathbf{r} \,.\, \mathbf{n} = d$, then

$$\sin\theta = \frac{\mathbf{b} \cdot \mathbf{n}}{|\mathbf{b}||\mathbf{n}|}$$

Cartesian Form If θ is the angle between the line

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

$a_2x + b_2y + c_2z + d = 0$, then

$$\sin\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

TRICK(S) FOR PROBLEM SOLVING

- If the line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ is parallel to the plane $a_2x + b_2y + c_2z + d = 0$, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

POINT OF INTERSECTION OF A LINE AND A PLANE

Working rule for finding the point of intersection of a line and a plane:

Step I: Write the coordinates of any point on the line in terms of some parameter r (say).

Step II: Substitute these coordinates in the equation of the plane to obtain the value of r.

Step III: Put the value of r in the coordinates of the point in step I.

TRICK(S) FOR PROBLEM SOLVING

The ratio in which the line segment PQ, joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. is divided by plane

$ax + by + cz + d = 0$ is, $-\left(\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}\right)$.

Planes Bisecting the Angles between Two Planes

Cartesian Form The equations of the planes bisecting the angles between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{(a_1x + b_1y + c_1z + d_1)}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{(a_2x + b_2y + c_2z + d_2)}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad (1)$$

Vector Form The equations of the planes bisecting the angles between the planes $\mathbf{r} \,.\, \mathbf{n}_1 = d_1$ and $\mathbf{r} \,.\, \mathbf{n}_2 = d_2$ are

$$\frac{|\mathbf{r}.\mathbf{n}_1 - \mathbf{d}_1|}{|\mathbf{n}_1|} = \frac{|\mathbf{r}.\mathbf{n}_2 - \mathbf{d}_2|}{|\mathbf{n}_2|}$$

or

$$\mathbf{r} \cdot (\hat{\mathbf{n}}_1 \pm \hat{\mathbf{n}}_2) = \frac{d_1}{|\mathbf{n}_1|} \pm \frac{d_2}{|\mathbf{n}_2|}$$

Bisector of the Angle Containing the Origin After making the constant term in both the equations positive, the positive sign in (1) gives the bisector of the angle which contains the origin.

Bisector of Acute/Obtuse Angle

(a) Write the equations of the given planes such that their constant terms are positive.

(b) If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, then origin lies in obtuse angle and hence positive sign in (1) gives the bisector of the obtuse angle.

(c) If $a_1a_2 + b_1b_2 + c_1c_2 < 0$, then origin lies in acute angle and hence positive sign in (1) gives the bisector of the acute angle.

Distance of a Point from a Plane

Vector Form The length of the perpendicular from a point having position vector **a** to the plane $\mathbf{r} . \mathbf{n} = d$ is given by,

$$p = \frac{|\mathbf{a}.\mathbf{n} - d|}{|\mathbf{n}|}$$

Cartesian Form The length of the perpendicular from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by,

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Distance between Two Parallel Planes

Vector Form The distance between two parallel planes $\mathbf{r} . \mathbf{n} = d_1$ and $\mathbf{r} . \mathbf{n} = d_2$ is given by,

$$p = \frac{|d_1 - d_2|}{|\mathbf{n}|}$$

Cartesian Form The distance between two parallel planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_1x + b_1y + c_1z + d_2 = 0$ is, given by,

$$p = \frac{|d_1 - d_2|}{\sqrt{a_1^2 + b_1^1 + c_1^2}}$$

Planes Passing through the Intersection of Two Planes

Vector Form The equation of a plane passing through the intersection of the planes $\mathbf{r} . \mathbf{n}_1 = d_1$ and $\mathbf{r} . \mathbf{n}_2 = d_2$ is

$$(\mathbf{r} . \mathbf{n}_1 - d_1) + k(\mathbf{r} . \mathbf{n}_2 - d_2) = 0$$

or $$\mathbf{r} . (\mathbf{n}_1 + k\mathbf{n}_2) = d_1 + kd_2,$$

where k is an arbitrary constant.

Cartesian Form The equation of a plane passing through the intersection of planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is $(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$, where k is an arbitrary constant.

Two Sides of a Plane

The two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ lie on the same side or the opposite sides of the plane $ax + by + cz + d = 0$ according as $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ have the same sign or the opposite signs.

Condition for a Line to Lie in a Plane

Vector Form If the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ lines in the plane $\mathbf{r} \cdot \mathbf{n} = d$, then

$$\mathbf{b} \cdot \mathbf{n} = 0 \text{ and } \mathbf{a} \cdot \mathbf{n} = d.$$

Cartesian Form If the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ lies in the plane $ax + by + cz + d = 0$, then

(a) $ax_1 + by_1 + cz_1 + d = 0$, and
(b) $al + bm + cn = 0$.

Condition for the Two Lines to be Intersecting (Coplanar) and the Equation of the Plane Containing Them

Vector Form If the lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$ are intersecting (coplanar), then

$$[\mathbf{a}_1\, \mathbf{b}_1\, \mathbf{b}_2] = [\mathbf{a}_2\, \mathbf{b}_1\, \mathbf{b}_2]$$

and the equation of the plane containing the two lines is

$$[\mathbf{r}\, \mathbf{b}_1\, \mathbf{b}_2] = [\mathbf{a}_1\, \mathbf{b}_1\, \mathbf{b}_2]$$

or $$[\mathbf{r}\, \mathbf{b}_1\, \mathbf{b}_2] = [\mathbf{a}_2\, \mathbf{b}_1\, \mathbf{b}_2]$$

Cartesian Form If the lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ are intersecting (coplanar)

then $$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and the equation of the plane containing the two lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or $$\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

IMAGE OF A POINT IN A PLANE

Let P and Q be two points and let π be a plane such that

(i) Line PQ is perpendicular to the plane π, and
(ii) Mid-point of PQ lies on the plane π.

Then either of the point is the image of the other in the plane π.

To find the image of a point in a given plane, we proceed as follows

(i) Write the equations of the line passing through P and normal to the given plane as

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

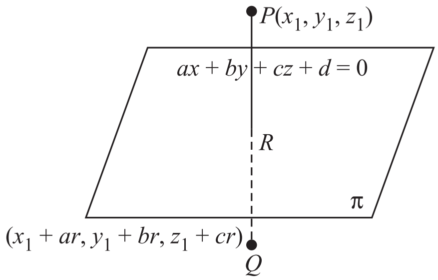


Fig. 22.11

(ii) Write the co-ofdinates of image Q as

$$(x_1 + ar, y_1 + br, z_1 + cr)$$

(iii) Find the co-ordinates of the mid-point R of PQ.

(iv) Obtain the value of r by putting the coordinates of R in the equation of the plane.

(v) Put the value of r in the coordinates of Q.

IMAGE OF A LINE ABOUT A PLANE

Let the line be $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and the plane be $a_2x + b_2y + c_2z + d = 0$

Find point of intersection (say P) of the line and the plane. Find image (say Q) of point (x_1, y_1, z_1) about the plane. Line PQ is the reflected line.

SOLVED EXAMPLES

12. The direction ratios of a normal to the plane through (1, 0, 0), (0, 1, 0), which makes an angle of $\frac{\pi}{4}$ with the plane $x + y = 3$ are

(A) $1, \sqrt{2}, 1$ (B) $1, 1, \sqrt{2}$

(C) 1, 1, 2 (D) $\sqrt{2}, 1, 1$

Solution: (B)

Any plane through (1, 0, 0) is

$$A(x - 1) + B(y - 0) + C(z - 0) = 0 \quad (1)$$

It contains (0, 1, 0) if $-A + B = 0$ (2)

Also, (1) makes an angle of $\frac{\pi}{4}$ with the plane $x + y = 3$, therefore, $\cos\frac{\pi}{4}=\frac{|A+B|}{\sqrt{A^2+B^2+C^2}\sqrt{1^2+1^2}}$

$$\Rightarrow (A + B)^2 = A^2 + B^2 + C^2 \Rightarrow 2AB = C^2 \quad (3)$$

From (2) and (3), $C^2 = 2A^2 \Rightarrow C = \pm\sqrt{2}A$

Hence $A : B : C :: A : A : \pm\sqrt{2}A$.

$\therefore$ Direction ratios are $1 : 1 : \pm\sqrt{2}$

13. The equation of the plane through the points (2, 3, 1) and (4, –5, 3) and parallel to x-axis is

(A) $x - z - 1 = 0$ (B) $4x + y - 11 = 0$

(C) $y + 4z - 7 = 0$ (D) none of these

Solution: (C)

Any plane parallel to x-axes is $by + cz + d = 0$.

If it passes through (2, 3, 1) and (4, –5, 3), then $3b + c + d = 0$ and $-5b + 3c + d = 0$,

i.e., $\frac{b}{1-3}=\frac{c}{-5-3}=\frac{d}{9+5}$ i.e., $\frac{b}{1}=\frac{c}{4}=\frac{d}{-7}$

Hence, the plane parallel to x-axis is $y + 4z - 7 = 0$.

14. The equation of the plane perpendicular to the yz-plane and passing through the points (1, –2, 4) and (3, –4, 5) is

(A) $y + 2z = 5$ (B) $2y + z = 5$

(C) $y + 2z = 6$ (D) $2y + z = 6$

Solution: (C)

Let the plane be

$$ax + by + cz + d = 0, \quad (1)$$

The yz-plane is $x = 0$ or $1.x + 0.y + 0.z = 0$. (2)

Since (1) and (2) are perpendicular to each other, we have

$$a \cdot 1 + b \cdot 0 + c \cdot 0 = 0, \text{ i.e., } a = 0.$$

$\therefore$ The plane (1) reduces to

$$by + cz + d = 0.$$

Now since it passes through (1, –2, 4) and (3, –4, 5), we get $-2b + 4c + d = 0$

and $-4b + 5c + d = 0$

giving $\frac{b}{-1}=\frac{c}{-2}=\frac{d}{6}$

Thus the plane is $y + 2z = 6$.

15. The point in which the line $\frac{x+1}{-1}=\frac{y-12}{5}=\frac{z-7}{-2}$ cuts the surface $11x^2 - 5y^2 - z^2 = 0$ is

(A) (2, –3, 1) (B) (2, 3, –1)

(C) (1, 2, 3) (D) (1, 2, –3)

Solution: (A, C)

Let $\frac{x+1}{-1}=\frac{y-12}{5}=\frac{z-7}{2}=r.$

Any point on the line is $(-r - 1, 5r + 12, 2r + 7)$ for every value of r.

If this point lies on the surface $11x^2 - 5y^2 + z^2 = 0$, then

$$11(-r - 1)^2 - 5(5r + 12)^2 + (2r + 7)^2 = 0$$

i.e., $110r^2 + 550r + 660 = 0$,

i.e., $r^2 + 5r + 6 = 0$

i.e., $(r + 3)(r + 2) = 0$, i.e., $r = -3, -2$

For these two values of r, the two points in which the given line cuts the surface are (2, –3, 1) and (1, 2, 3).

16. The equation of the plane through the line $x + y + z + 3 = 0 = 2x - y + 3z + 1$ and parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is

(A) $x - 5y + 3z = 7$ (B) $x - 5y + 3z = -7$
(C) $x + 5y + 3z = 7$ (D) $x + 5y + 3z = -7$

Solution: (A)

Any plane through the given line

$$2x - y + 3z + 1 + \lambda(x + y + z + 3) = 0$$

(From $S + \lambda S' = 0$)

If this plane is parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$, then the normal to the plane is also perpendicular to the above line or

$$(2 + \lambda)1 + (\lambda - 1)2 + (3 + \lambda)3 = 0$$

(From $l_1l_2 + m_1m_2 + n_1n_2 = 0$)

This gives $\lambda = -\frac{3}{2}$ and the required plane is

$$x - 5y + 3z - 7 = 0$$

17. The equation of the plane containing the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and the point (0, 7, –7) is

(A) $x + y + z = 2$ (B) $x + y + z = 3$
(C) $x + y + z = 0$ (D) none of these

Solution: (C)

Any plane containing $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ is

$$a(x + 1) + b(y - 3) + c(z + 2) = 0 \quad (1)$$

where $$-3a + 2b + c = 0 \quad (2)$$

If the plane through (0, 7, –7), then

$$a + 4b - 5c = 0 \quad (3)$$

From (2) and (3), $\frac{a}{-10-4} = \frac{b}{1-15} = \frac{c}{-12-2}$,

i.e., $\frac{a}{1} = \frac{b}{1} = \frac{c}{1}$

Hence the plane (1) becomes

$$(x + 1) + (y - 3) + (z + 2) = 0, \text{ i.e., } x + y + z = 0$$

18. The equation of the plane passing through the straight line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ and perpendicular to the plane $x + 2y + z = 12$ is

(A) $9x + 2y - 5z + 4 = 0$
(B) $9x - 2y - 5z + 4 = 0$
(C) $9x + 2y + 5z + 4 = 0$
(D) none of these

Solution: (B)

Any plane through the given line is

$$a(x - 1) + b(y + 1) + c(z - 3) = 0 \quad (1)$$

where $$2a - b + 4c = 0 \quad (2)$$

If this plane is perpendicular to $x + 2y + z = 12$, then their normals are also perpendicular to each other.

$\therefore$ $$a + 2b + c = 0 \quad (3)$$

From (2) and (3), $\frac{a}{-1-8} = \frac{b}{4-2} = \frac{c}{4+1}$,

i.e., $\frac{a}{-9} = \frac{b}{2} = \frac{c}{5}$

$\therefore$ plane (1) becomes

$$-9(x - 1) + 2(y + 1) + 5(z - 3) = 0$$

i.e., $$9x - 2y - 5z + 4 = 0$$

19. The position vectors of points A and B are $\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} + 3\hat{j} + 3\hat{k}$ respectively. The equation of a plane is $\mathbf{r}.(5\hat{i} + 2\hat{j} - 7\hat{k}) + 9 = 0.$ The points A and B

(A) lie on the plane
(B) are on the same side of the plane
(C) are on the opposite side of the plane
(D) none of these

Solution: (C)

The position vectors of two given points are $\mathbf{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\mathbf{b} = 3\hat{i} + 3\hat{j} + 3\hat{k}$ and the equation of the given plane is $\mathbf{r} \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9 = 0$ or $\mathbf{r} \cdot \mathbf{n} + \mathbf{d} = 0$.

We have, $\mathbf{a} \cdot \mathbf{n} + \mathbf{d} = (\hat{i} - \hat{j} - 3\hat{k}) \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9$
$= 5 - 2 - 21 + 9 < 0$

and $\mathbf{b} \cdot \mathbf{n} + d = (3\hat{i} + 3\hat{j} - 3\hat{k}) \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9$
$= 15 + 6 - 21 + 9 > 0$

So, the points **a** and **b** are on the opposite sides of the plane.

20. Lines $\mathbf{r} = \mathbf{a}_1 + t\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + s\mathbf{b}_2$ lie on a plane if

(A) $\mathbf{a}_1 \times \mathbf{a}_2 = \mathbf{O}$ (B) $\mathbf{b}_1 \times \mathbf{b}_2 = \mathbf{O}$
(C) $(\mathbf{a}_2 - \mathbf{a}_1) . (\mathbf{b}_1 \times \mathbf{b}_2) = \mathbf{O}$ (D) none of these

Solution: (C)

Lines lie in a plane if

$$(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = 0$$

$\because$ $\mathbf{b}_1 \times \mathbf{b}_2$ is a vector $\perp$ to $\mathbf{b}_1$, $\mathbf{b}_2$.

21. A square $ABCD$ of diagonal $2a$ is folded along the diagonal AC so that the planes DAC and BAC are at right angle. The shortest distance between DC and AB is

(A) $\sqrt{2}a$ (B) $\frac{2a}{\sqrt{3}}$
(C) $\frac{2a}{\sqrt{5}}$ (D) $\left(\frac{\sqrt{3}}{2}\right)a$

Solution: (B)

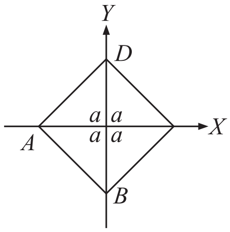
When folded coordinates will be $D(0, 0, a)$; $C(a, 0, 0)$; $A(-a, 0, 0)$; $B(0, -a, 0)$

Equation of DC is,

$$\frac{x}{a} = \frac{y}{0} = \frac{z-a}{-a}$$

Equation of AB is,

$$\frac{x+a}{a} = \frac{y}{-a} = \frac{z}{0}$$



$\therefore$ Shortest distance $= \dfrac{2a}{\sqrt{3}}$.

22. The equation of the plane containing the line $\mathbf{r} = \hat{i} + \hat{j} + t(2\hat{i} + \hat{j} + 4\hat{k})$, is

(A) $\mathbf{r}\cdot(\hat{i}+2\hat{j}-\hat{k}) = 3$

(B) $\mathbf{r}\cdot(\hat{i}+2\hat{j}-\hat{k}) = 6$

(C) $\mathbf{r}\cdot(-\hat{i}-2\hat{j}+\hat{k}) = 3$

(D) none of these

Solution: (A)

The position vector of any point on the given line is $\hat{i} + \hat{j} + t(2\hat{i} + \hat{j} + 4\hat{k}) = (1+2t)\hat{i} + (1+t)\hat{j} + 4t\hat{k}$

This lies on $\mathbf{r}\cdot(\hat{i}+2\hat{j}-\hat{k}) = 3$

if $(1 + 2t)\cdot 1 + (1 + t)\cdot 2 + 4t(-1) = 3$

i.e., if $1 + 2t + 2 + 2t - 4t = 3$. i.e., if $3 = 3$ which is true.

Hence, the plane $\mathbf{r}\cdot(\hat{i}+2\hat{j}-\hat{k}) = 3$ contains the given line.

23. The line of intersection of the planes $\mathbf{r}\cdot(3\hat{i}-\hat{j}+\hat{k}) = 1$ and $\mathbf{r}\cdot(\hat{i}+4\hat{j}-2\hat{k}) = 2$ is parallel to the vector

(A) $-2\hat{i}+7\hat{j}+13\hat{k}$ (B) $2\hat{i}+7\hat{j}-13\hat{k}$

(C) $-2\hat{i}-7\hat{j}+13\hat{k}$ (D) $2\hat{i}+7\hat{j}+13\hat{k}$

Solution: (A)

The line of intersection of the planes

$\mathbf{r}\cdot(3\hat{i}-\hat{j}+\hat{k}) = 1$ and

$\mathbf{r}\cdot(\hat{i}+4\hat{j}-2\hat{k}) = 2$ is $\perp$ to each of the normal vectors

$$\mathbf{n}_1 = 3\hat{i}-\hat{j}+\hat{k} \text{ and } n_2 = \hat{i}+4\hat{j}-2\hat{k}$$

$\therefore$ It is parallel to the vector

$$\mathbf{n}_1 \times \mathbf{n}_2 = (3\hat{i}-\hat{j}+\hat{k})\times(\hat{i}+4\hat{j}-2\hat{k})$$

$$= -2\hat{i}+7\hat{j}+13\hat{k}$$

24. The distance between the planes $\mathbf{r}\cdot(2\hat{i}-\hat{j}+3\hat{k}) = 4$ and $\mathbf{r}\cdot(6\hat{i}-3\hat{j}+9\hat{k})+13 = 0$ is

(A) $\dfrac{5}{3\sqrt{14}}$ (B) $\dfrac{10}{3\sqrt{14}}$

(C) $\dfrac{25}{3\sqrt{14}}$ (D) none of these

Solution: (C)

The distance between the parallel planes

$$\mathbf{r}\cdot(2\hat{i}-\hat{j}+3\hat{k}) = 4$$

and $$\mathbf{r}\cdot(2\hat{i}-\hat{j}+3\hat{k}) = -\frac{13}{3}$$

$$\left[\because \begin{array}{l} \mathbf{r}\cdot(6\hat{i}-3\hat{j}+9\hat{k})+13 = 0 \\ \text{so } \mathbf{r}\cdot(2\hat{i}-\hat{j}+3\hat{k}) + -\dfrac{13}{3} \end{array}\right]$$

is $\dfrac{\left|4-\left(\dfrac{-13}{3}\right)\right|}{\sqrt{2^2+(-1)^2+3^2}}$ $\left[\because \text{ Reqd. distance} = \dfrac{|d-k|}{|\mathbf{n}|}\right]$

$$= \frac{\left|4+\frac{13}{3}\right|}{\sqrt{4+1+9}} = \frac{\frac{25}{3}}{\sqrt{14}} = \frac{25}{3\sqrt{14}}.$$

25. The ratio in which the plane $r\cdot(\hat{i}-2\hat{j}+2\hat{k}) = 17$ divides the line joining the points $-2\hat{i}+4\hat{j}+7\hat{k}$ and $3\hat{i}-5\hat{j}+8\hat{k}$ is

(A) 3 : 5 (B) 1 : 10

(C) 3 : 10 (D) 1 : 5

Solution: (C)

Let the plane $\mathbf{r}\cdot(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 17$ divide the line joining the points

$-2\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$ and $3\mathbf{i} - 5\mathbf{j} + 8\mathbf{k}$ in the ratio $t : 1$ at the point P.

$\therefore$ P is $\dfrac{3t-2}{t+1}i + \dfrac{-5t+4}{t+1}j + \dfrac{8t+7}{t+1}k$

This lies on the given plane,

$\therefore$ $\dfrac{3t-2}{t+1}\cdot 1 + \dfrac{-5t+4}{t+1}(-2) + \dfrac{8t+7}{t+1}(3) = 17$

$\Rightarrow$ $3t - 2 + 10t - 8 + 24t + 21 = 17t + 17$

$\therefore$ $20t = 17 - 21 + 10 = 6 \Rightarrow t = \dfrac{6}{20} = \dfrac{3}{10}$

$\therefore$ Reqd. ratio is 3 : 10.

26. A plane is parallel to lines whose direction ratios are (1, 0, –1) and (–1, 1, 0) and it contains the point (1, 1, 1). If it cuts coordinates axes at A, B, C then the volume of the tetrahedron $OABC$ is

(A) 9/5 cu units (B) 9/4 cu units

(C) 9/2 cu units (D) none of these

Solution: (C)

Let the equation of the plane through (1, 1, 1) be

$$a(x-1) + b(y-1) + c(z-1) = 0$$

Since it is parallel to the straight lines having dr's (1, 0, –1) and (–1, 1, 0), therefore

$$a - c = 0 \text{ and } -a + b = 0$$
$\Rightarrow a = b = c$

Therefore, equation of plane is $x - 1 + y - 1 + z - 1 = 0$

or $$\frac{x}{3} + \frac{y}{3} + \frac{z}{3} = 1$$

Its intercepts on coordinate axes are $A(3, 0, 0)$, $B(0, 3, 0)$ and $C(0, 0, 3)$. Hence, the voluem of tetrahedrron $OABC$.

$$= \frac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$
$$= \frac{1}{6}\begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix} = \frac{27}{6} = \frac{9}{2} \text{cu. units.}$$

27. Gives the line $L: \frac{x-1}{3} = \frac{y+1}{2} = \frac{z-3}{-1}$ and the plane π : $x - 2y = 0$. Of the following assertions, the only one that is always true is

(A) L is $\perp$ to π (B) L lies in π
(C) L is parallel to π (D) none of these

Solution: (B)

Since $3(1) + 2(-2) + (-1)(-1) = 3 - 4 + 1 = 0$,
$\therefore$ given line is $\perp$ to the normal to the plane i.e., given line is parallel to the given plane.
Also $(1, -1, 3)$ lies on the plane $x - 2y - z = 0$ if
$$1 - 2(-1) - 3 = 0 \text{ i.e., } 1 + 2 - 3 = 0$$
which is true $\therefore$ L lies in plane π.

28. The distance between the line
$\mathbf{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ and the plane
$\mathbf{r}\cdot(\hat{i} + 5\hat{j} + \hat{k}) = 5$ is

(A) $\frac{10}{9}$ (B) $\frac{10}{3\sqrt{3}}$
(C) $\frac{10}{3}$ (D) none of these

Solution: (B)

The given line is $\mathbf{r} = \mathbf{a} + t\mathbf{b}$
where $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and given plane is
$$\mathbf{r} \cdot \mathbf{n} = \mathbf{p},$$
where $$n = \hat{i} + 5\hat{j} + \hat{k}, p = 5$$
Since $$\mathbf{b} \cdot \mathbf{n} = 1 - 5 + 4 = 0$$
$\therefore$ given line is parallel to the given plane $\therefore$ the distance between the line and the plane is equal to length of the perpendicular from the point $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ on the line to the given plane.

$$\therefore \text{Reqd. distance } = \left|\frac{(2\mathbf{i}-2\mathbf{j}+3\mathbf{k})\cdot(\mathbf{i}+5\mathbf{j}+\mathbf{k})-5}{\sqrt{1+25+1}}\right|$$
$$= \left|\frac{2-10+3-5}{\sqrt{27}}\right| = \frac{10}{3\sqrt{3}}.$$

29. The equation of the plane containing the two lines $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{3}$ and $\frac{x}{2} = \frac{y-2}{-1} = \frac{z+1}{3}$ is

(A) $8x + y - 5z - 7 = 0$
(B) $8x + y + 5z - 7 = 0$
(C) $8x - y - 5z - 7 = 0$
(D) none of these

Solution: (A)

Equation of any plane through the first line is
$$a(x - 1) + b(y + 1) + cz = 0 \quad (1)$$
where $$2a - b + 3c = 0 \quad (2)$$
It will pass through the second line if $(0, 2, -1)$, a point on the second line lies on it
i.e., if $$-a + 3b - c = 0 \quad (3)$$
Solving (2) and (3), we get
$$\frac{a}{-8} = \frac{b}{-1} = \frac{c}{5} \text{ or } \frac{a}{8} = \frac{b}{1} = \frac{c}{-5}$$
Hence equation of required plane is
$8(x - 1) + (y + 1) - 5z = 0 \Rightarrow 8x + y - 5z - 7 = 0.$

30. If the straight lines $x = 1 + s$, $y = -3 - \lambda s$, $z = 1 + \lambda s$ and $x = \frac{t}{2}, y = 1 + t, z = 2 - t$ with parameters s and t respectively, are co-planar, then λ equals

(A) 0 (B) -1
(C) $-\frac{1}{2}$ (D) -2

Solution: (D)

The given lines are
$$x - 1 = \frac{y+3}{-\lambda} = \frac{z-1}{\lambda} = s \quad (1)$$
and $$2x = y - 1 = \frac{z-2}{-1} = t \quad (2)$$
The lines are coplanar, if
$$\begin{vmatrix} 0-(-1) & -1-3 & -2-(-1) \\ 1 & -\lambda & \lambda \\ \frac{1}{2} & 1 & -1 \end{vmatrix} = 0$$
Operate $C_2 \to C_2 + C_3$
$$\begin{vmatrix} 1 & -5 & -1 \\ 1 & 0 & \lambda \\ \frac{1}{2} & 0 & -1 \end{vmatrix} = 0$$
$$\Rightarrow 5\left(-1 - \frac{\lambda}{2}\right) = 0 \Rightarrow \lambda = -2$$

31. A straight line $\vec{r} = \vec{a} + \lambda\vec{b}$ meets the ;lane $\vec{r}.\vec{n} = 0$ in P. The position vector of P is

(A) $\vec{a} + \dfrac{\vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}\vec{b}$ (B) $\vec{a} - \dfrac{\vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}\vec{b}$

(B) $\vec{a} - \dfrac{\vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}\vec{b}$ (D) none of these

Solution: (C)

The straight line $\vec{r} = \vec{a} + \lambda\vec{b}$ meets the plane $\vec{r}\cdot\vec{n} = 0$ in P for which λ is given by

$$(\vec{a} + \lambda\vec{b})\cdot\vec{n} = 0 \quad \Rightarrow \quad \lambda = -\frac{\vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}$$

Thus, the positive vector of P is

$$\vec{r} = \vec{a} - \frac{\vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}\vec{b}$$

SPHERE

A sphere is the locus of a point which remains at a constant distance from a fixed point. The constant distance is called the radius and the fixed point is called the centre of the sphere.

EQUATION OF A SPHERE

Vector Equation The vector equation of a sphere of radius a and centre having position vector **c** is $|\mathbf{r} - \mathbf{c}| = a$

IMPORTANT POINTS

The vector equation of a sphere of radius a with centre at the origin, is $|\mathbf{r}| = a$.

Cartesian Equation The equation of a sphere with centre (a, b, c) and radius k is given by

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = k^2$$

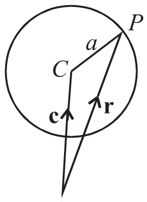


Fig. 22.12

IMPORTANT POINTS

The equation of a sphere with centre at origin and radius k is

$$x^2 + y^2 + z^2 = k^2.$$

General Equation of a Sphere

The general equation

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

represents a sphere with centre $(-u, -v, -w)$ and radius equal to $\sqrt{u^2 + v^2 + w^2 - d}$

Equation of a Sphere through Four Points

Equation of a sphere passing through four non-coplanar points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

or

(a) Assume the equation of the sphere to be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad (1)$$

(b) Put the coordinates of four given points in Eqn. (1) to obtain four equations in u, v, w and d.

(c) Solve the four equations obtained in Step (b) to get the values of u, v, w, and d.

(d) Put the values of u, v, w and d obtained in Step (c) in Eqn. (1) to obtain the required equation of sphere.

Equation of a Sphere, the Extremities of Diameter Being given

Cartesian Form The equation of a sphere described on the join of two points

$$P(x_1, y_1, z_1) \text{ and } Q(x_2, y_2, z_2)$$

as diameter is given by

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0.$$

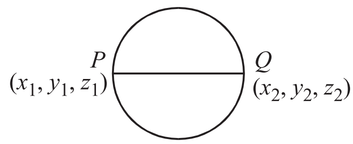


Fig. 22.13

Vector Form The vector equation of a sphere, described on the join of two points P and Q, having position vectors **a** and **b**, as diameter, is given by

$$(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$$

or $\quad |\mathbf{r}^2| - \mathbf{r} \cdot (\mathbf{a} - \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} = 0$

or $\quad |\mathbf{r} - \mathbf{a}|^2 + |\mathbf{r} - \mathbf{b}|^2 = |\mathbf{a} - \mathbf{b}|^2$

Condition of Tangency

Vector Form Condition for the plane $\mathbf{r} \cdot \mathbf{n} = d$ to touch the sphere $|\mathbf{r} - \mathbf{c}| = a$ is

$$\frac{|\mathbf{c} \cdot \mathbf{n} - d|}{|\mathbf{n}|} = a$$

Cartesian Form Condition for the plane $lx + my + nz = p$ to touch the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is $(ul + vm + wn + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$.

SECTION OF A SPHERE BY A PLANE

Consider a sphere intersected by a plane. The set of points common to both sphere and plane is called a *plane section* of a sphere. The plane section of a sphere is always a circle. The equations of the sphere and the plane taken together represent the plane section.

Let C be the centre of the sphere and M be the foot of the perpendicular from C on the plane. Then M is the centre of the circle and radius of the circle is given by $PM = \sqrt{CP^2 - CM^2}$.

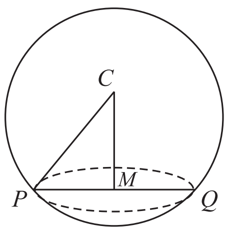


Fig. 22.14

The centre M of the circle is the point of intersection of the plane and line CM which passes through C and is perpendicular to the given plane.

Centre: The foot of the perpendicular from the centre of the sphere to the plane is the centre of the circle.

(Radius of circle)^2 = (Radius of sphere)^2 – (Perpendicular from centre of sphere on the plane)^2

Great Circle: The section of a sphere by a plane through the centre of the sphere is a great circle. Its centre and radius are the same as those of the given sphere.

SOLVED EXAMPLES

32. The radius of the circular section of the sphere $|\mathbf{r}| = 5$ by the plane $\mathbf{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 3\sqrt{3}$ is

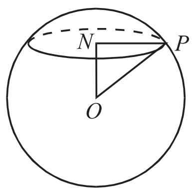
(A) 16 (B) 8
(C) 4 (D) none of these

Solution: (C)

The sphere $|\mathbf{r}| = 5$ has centre at the origin and radius 5. Distance of the plane

$\mathbf{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 3\sqrt{3}$ from the origin.

$$= \frac{3\sqrt{3}}{|\hat{i} + \hat{j} + \hat{k}|} = \frac{3\sqrt{3}}{\sqrt{1^2 + 1^2 + 1^2}} = 3.$$



Thus, in figure

$OP = 5$, $ON = 3$.

$\therefore \quad NP^2 = OP^2 - ON^2 = (5)^2 - (3)^2 = 16, \quad \therefore NP = 4$

Hence, the radius of the circular section = $NP = 4$.

33. The equation of the sphere whose centre has the position vector $(3\hat{i} + 6\hat{j} - 4\hat{k})$ and which touches the plane $\mathbf{r} \cdot (2\hat{i} + 2\hat{j} + \hat{k})| = 10$ is

(A) $|\mathbf{r} - (3\hat{i} + 6\hat{j} - 4\hat{k})| = 4$

(B) $|\mathbf{r} - (3\hat{i} + 6\hat{j} + 4\hat{k})| = 4$

(C) $|\mathbf{r} - (3\hat{i} + 6\hat{j} - 4\hat{k})| = 2$

(D) none of these

Solution: (A)

The centre of the sphere has the position vector

$$3\hat{i} + 6\hat{j} - 4\hat{k}.$$

Radius = The distance of the centre whose position vector

$\mathbf{a} = 3\hat{i} + 6\hat{j} - 4\hat{k}$ from the plane

$\mathbf{r} \cdot (2\hat{i} + 2\hat{j} + \hat{k})| = 10$

$$= \frac{|\mathbf{a} \cdot \mathbf{n} - d|}{|\mathbf{n}|}$$

$$= \frac{|(3\hat{i} + 6\hat{j} - 4\hat{k}) \cdot (2\hat{i} - 2\hat{j} - \hat{k}) - 10|}{\sqrt{2^2 + (-2)^2 + (-1)^2}}$$

$$= \frac{|6 - 12 + 4 - 10|}{\sqrt{4 + 4 + 1}} = \frac{|-12|}{3} = \frac{12}{3} = 4$$

Centre = (3, 6, –4); Radius = 4.

Required equation of the sphere is

$$|\mathbf{r} - (3\hat{i} + 6\hat{j} - 4\hat{k})| = 4$$

34. The equation $|\mathbf{r}|^2 - \mathbf{r} \cdot (2\hat{i} + 4\hat{j} - 2\hat{k}) - 10 = 0$ represents a

(A) circle (B) plane
(C) sphere of radius 4 (D) sphere of radius 3
(E) none of these

Solution: (C)

Since the equation $|\mathbf{r}|^2 - 2(\mathbf{r}\cdot\mathbf{a}) + \lambda = 0$ represents a sphere of radius $\sqrt{|\mathbf{a}|^2 - \lambda}$, therefore

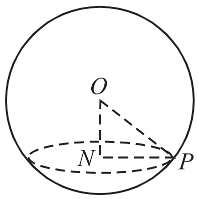
$|\mathbf{r}|^2 - \mathbf{r}\cdot(2\hat{i}+4\hat{j}-2\hat{k}) - 10 = 0$ represents a sphere of radius $= \sqrt{|\hat{i}+2\hat{j}-\hat{k}|^2 + 10} = \sqrt{6+10} = 4.$

35. The radius of the circle $x^2 + y^2 + z^2 = 49$, $2x + 3y - z - 5\sqrt{14} = 0$ is

(A) $\sqrt{6}$ (B) $2\sqrt{6}$
(C) $4\sqrt{6}$ (D) none of these

Solution: (B)

The sphere $x^2 + y^2 + z^2 = 49$ has centre at the origin (0, 0, 0) and radius 7.



Distance of the plane $2x + 3y - z - 5\sqrt{14} = 0$ from the origin.

$$= \frac{|2(0)+3(0)-(0)-5\sqrt{14}|}{\sqrt{2^2+3^2+(-1)^2}} = \frac{|-5\sqrt{14}|}{\sqrt{14}} = \frac{5\sqrt{14}}{\sqrt{14}} = 5.$$

Thus in Fig.

$OP = 7$, $ON = 5$

$NP^2 = OP^2 - ON^2 = (7)^2 - (5)^2 = 49 - 25 = 24$

$\therefore\ NP = 2\sqrt{6}.$

Hence the radius of the circle $= NP = 2\sqrt{6}.$

36. The smallest radius of the sphere passing through (1, 0, 0), (0, 1, 0) and (0, 0, 1) is

(A) $\sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{8}}$
(C) $\sqrt{\frac{5}{6}}$ (D) $\sqrt{\frac{5}{12}}$

Solution: (A)

Let the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

It passes through (1, 0, 0) $\Rightarrow 1 + 2u + d = 0$ (1)

It passes through (0, 1, 0) $\Rightarrow 1 + 2v + d = 0$ (2)

It passes through (0, 0, 1) $\Rightarrow 1 + 2w + d = 0$ (3)

$\therefore\ u = v = w = -\frac{1+d}{2}$

Radius of the sphere $= \sqrt{u^2+v^2+w^2-d}$

$$= \frac{1}{2}\sqrt{3(1+d)^2 - 4d}$$

$$= \frac{1}{2}\sqrt{3d^2+2d+3}$$

Now $3d^2 + 2d + 3 \geq \frac{4\times3\times3-(2)^2}{4\times3} = \frac{32}{12} = \frac{8}{3}$

$\therefore$ radius $\geq \frac{1}{2}\times\sqrt{\frac{8}{3}} = \sqrt{\frac{2}{3}}$

37. Radius of the circle $\mathbf{r}^2 + \mathbf{r}\cdot(2\hat{i}-2\hat{j}-4\hat{k}) - 19 = 0$ $\mathbf{r}\cdot(\hat{i}-2\hat{j}+4\hat{k}) + 8 = 0$

(A) 5 (B) 4
(C) 3 (D) 2

Solution: (B)

Given circle is intersection of sphere

$x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0$ (1)

and plane $x - 2y + 2z + 8 = 0$ (2)

Centre of sphere is (–1, 1, 2).

p = Length of the $\perp$ from, (–1, 1, 2) upon (2)

$$= \frac{-1-2+4+8}{\sqrt{1+4+4}} = \frac{9}{3} = 3$$

R = Radius of the sphere $= \sqrt{1+1+4+19} = 5$

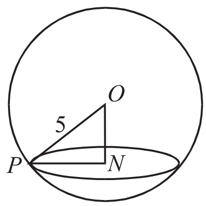
Radius of the circle $= \sqrt{R^2 - p^2} = \sqrt{25-9} = \sqrt{16} = 4.$

38. The position vector of the centre of the circle $|\mathbf{r}| = 5, \mathbf{r}\cdot(\hat{i}+\hat{j}+\hat{k}) = 3\sqrt{3}$ is

(A) $\sqrt{3}(\hat{i}+\hat{j}+\hat{k})$ (B) $\hat{i}+\hat{j}+\hat{k}$
(C) $3(\hat{i}+\hat{j}+\hat{k})$ (D) none of the above

Solution: (A)

The equation of ON is $\mathbf{r} = \lambda(\mathbf{i}+\mathbf{j}+\mathbf{k})$ (1)



Since it passes through the origin and is parallel to the vector $(\mathbf{i}+\mathbf{j}+\mathbf{k})$, any pt. on it is $\lambda(\mathbf{i}+\mathbf{j}+\mathbf{k})$. If this pt. lies on the plane $r\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k}) = 3\sqrt{3}$

then $\lambda(\mathbf{i}+\mathbf{j}+\mathbf{k})\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = 3\sqrt{3}$

or $\lambda(1+1+1) = 3\sqrt{3}$

$\therefore\ \lambda = \sqrt{3}$

Putting the value of λ in (i), we get the position vector N i.e., centre of the circle as $\sqrt{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})$.

39. The coordinates of a point which is equidistant from the points (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c) are given by

(A) $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ (B) $\left(\frac{-a}{2}, \frac{-b}{2}, \frac{c}{2}\right)$

(C) $\left(\frac{a}{2}, \frac{-b}{2}, \frac{-c}{2}\right)$ (D) $\left(\frac{-a}{2}, \frac{b}{2}, \frac{-c}{2}\right)$

Solution: (A)

Sphere passing through $(a, 0, 0)$ $(0, b, 0)$ $(0, 0, c)$ and $(0, 0, 0)$ is $x^2 + y^2 + z^2 - ax - by - cz = 0$. Its centre $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ is equidistant from given points.

40. A plane passes through a fixed point (a, b, c). The locus of the foot of the perpendicular to it from the origin is a sphere of radius

(A) $\sqrt{a^2 + b^2 + c^2}$ (B) $\frac{1}{2}\sqrt{a^2 + b^2 + c^2}$

(C) $a^2 + b^2 + c^2$ (D) none of these

Solution: (B)

Let the foot of the perpendicular from the origin on the given plane be $P(\alpha, \beta, \gamma)$. Since, the plane passes through $A(a, b, c)$.

$\therefore\ AP \perp OP \Rightarrow \alpha(\alpha - a) + \beta(\beta - b) + \gamma(\gamma - c) = 0$

Hence, the locus of $(\alpha, \beta, \gamma)$is

$$x(x - a) + y(y - b) + z(z - c) = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - ax - by - cz = 0,$$

which is a sphere of radius $\frac{1}{2}\sqrt{a^2 + b^2 + c^2}$

EXERCISES

Single Option Correct Type

1. The equation of the plane through the points (2, 3, 1) and (4, –5, 3) and parallel to x-axis is

(A) $x - z - 1 = 0$ (B) $4x + y - 11 = 0$

(C) $y + 4z - 7 = 0$ (D) none of these

2. The edge of a cube is of length 'a' then the shortest distance between the diagonal of a cube and an edge skew to it is

(A) $a\sqrt{2}$ (B) a

(C) $\sqrt{2}/a$ (D) $a/\sqrt{2}$

3. A square $ABCD$ of diagonal $2a$ is folded along the diagonal AC so that the planes DAC and BAC are at right angle. The shortest distance between DC and AB is

(A) $\sqrt{2}a$ (B) $2a/\sqrt{3}$

(C) $2a/\sqrt{5}$ (D) $(\sqrt{3}/2)a$

4. The line of intersection of the planes $\mathbf{r} \cdot (3\mathbf{i} - \mathbf{j} + \mathbf{k}) = 1$ and $\mathbf{r} \cdot (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 2$ is parallel to the vector

(A) $-2\mathbf{i} + 7\mathbf{j} + 13\mathbf{k}$

(B) $2\mathbf{i} + 7\mathbf{j} - 13\mathbf{k}$

(C) $-2\mathbf{i} - 7\mathbf{j} + 13\mathbf{k}$

(D) $2\mathbf{i} + 7\mathbf{j} + 13\mathbf{k}$

5. The smallest radius of the sphere passing through (1, 0, 0), (0, 1, 0) and (0, 0, 1) is

(A) $\sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{8}}$

(C) $\sqrt{\frac{5}{6}}$ (D) $\sqrt{\frac{5}{12}}$

6. The position vector of the centre of the circle $|\mathbf{r}| = 5$, $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 3\sqrt{3}$ is

(A) $\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$ (B) $\mathbf{i} + \mathbf{j} + \mathbf{k}$

(C) $3(\mathbf{i} + \mathbf{j} + \mathbf{k})$ (D) none of the above

7. Perpendicular distance of the point (3, 4, 5) from the y-axis, is

(A) $\sqrt{34}$ (B) $\sqrt{41}$

(C) 4 (D) 5

8. A plane passes through a fixed point (a, b, c). The locus of the foot of the perpendicular to it from the origin is a sphere of radius

(A) $\sqrt{a^2 + b^2 + c^2}$

(B) $\frac{1}{2}\sqrt{a^2 + b^2 + c^2}$

(C) $a^2 + b^2 + c^2$

(D) none of these

9. The direction ratios of the line

$x - y + z - 5 = 0 = x - 3y - 6$ are

(A) 3, 1, –2 (B) 2, –4, 1

(C) $\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}$ (D) $\frac{2}{\sqrt{41}}, \frac{-4}{\sqrt{41}}, \frac{1}{\sqrt{41}}$

10. A straight line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ meets the p lane $\mathbf{r} \cdot \mathbf{n} = 0$ in P. The position vector of P is

(A) $\mathbf{a} + \frac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}\mathbf{b}$ (B) $\mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}\mathbf{b}$

(C) $\mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}\mathbf{b}$ (D) none of these

11. From the point (1, –2, 3), lines are drawn to meet the sphere $x^2 + y^2 + z^2 = 4$ and they are divided internally in the ratio 2 : 3. The locus of the point of division is

(A) $5x^2 + 5y^2 + 5z^2 - 6x + 12y + 2z = 0$
(B) $5(x^2 + y^2 + z^2) = 22$
(C) $5x^2 + 5y^2 + 5z^2 - 2xy - 3yz - zx - 6x + 12y + 5z + 22 = 0$
(D) none of these

12. The length of the perpendicular from the origin to the plane passing though three non-collinear points **a**, **b**, **c** is

(A) $\dfrac{[\mathbf{a\,b\,c}]}{|\mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{a}+\mathbf{b}\times\mathbf{c}|}$

(B) $\dfrac{2[\mathbf{a\,b\,c}]}{|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|}$

(C) $[\mathbf{a\ b\ c}]$
(D) none of these

13. The lines $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b}\times\mathbf{c})$ and $\mathbf{r} = \mathbf{b} + \mu(\mathbf{c}\times\mathbf{a})$ will intersect if

(A) $\mathbf{a}\times\mathbf{c} = \mathbf{b}\times\mathbf{c}$ (B) $\mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c}$
(C) $\mathbf{b}\times\mathbf{a} = \mathbf{c}\times\mathbf{a}$ (D) none of these

14. The length of the perpendicular from the origin to the plane passing through the point **a** and containing the line $\mathbf{r} = \mathbf{b} + \lambda\mathbf{c}$ is

(A) $\dfrac{[\mathbf{a\,b\,c}]}{|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|}$ (B) $\dfrac{[\mathbf{a\,b\,c}]}{|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}|}$

(C) $\dfrac{[\mathbf{a\,b\,c}]}{|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|}$ (D) $\dfrac{[\mathbf{a\,b\,c}]}{|\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}|}$

15. The equation of the plane which contains the origin and the line of intersection of the planes $\mathbf{r}.\mathbf{a} = p$ and $\mathbf{r}.\mathbf{b} = q$ is

(A) $\mathbf{r}\cdot(p\mathbf{a} - q\mathbf{b}) = 0$ (B) $\mathbf{r}\cdot(p\mathbf{a} + q\mathbf{b}) = 0$
(C) $\mathbf{r}\cdot(q\mathbf{a} + p\mathbf{b}) = 0$ (D) $\mathbf{r}\cdot(q\mathbf{a} - p\mathbf{b}) = 0$

16. The vector equation of the line of intersection of the planes $\mathbf{r}\cdot(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 0$ and $\mathbf{r}\cdot(3\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 0$ is

(A) $\mathbf{r} = \lambda(\mathbf{i} + 2\mathbf{i} + \mathbf{k})$ (B) $\mathbf{r} = \lambda(\mathbf{i} - 2\mathbf{i} + \mathbf{k})$
(C) $\mathbf{r} = \lambda(\mathbf{i} + 2\mathbf{i} - 3\mathbf{k})$ (D) none of these

17. The plane $x + y + z = 5\sqrt{3}$ and sphere $x^2 + y^2 + z^2 = 5$

(A) touch each other (B) cut in a circle
(C) do not meet (D) none of these

18. If $P(x, y, z)$ is a point on the line segment joining $Q(2, 2, 4)$ and $R(3, 5, 6)$ such that the projection of $\overrightarrow{OP}$ on the axes are $\frac{13}{5}, \frac{19}{5}, \frac{26}{5}$ respectively, then P divides QR in the ratio

(A) 1 : 2 (B) 3 : 2
(C) 2 : 3 (D) 1 : 3

19. From the point $P(a, b, c)$ the normals drawn to planes yz and zx are PA, PB, then the equation of plane OAB is

(A) $bcx + acy + abz = 0$
(B) $bcx + acy - abz = 0$
(C) $bcx - acy + abz = 0$
(D) $-bcx + acy + abz = 0$

20. A mirror and a source of light are situated at the origin O and at a point on OX, respectively. A ray of light from the source strikes the mirror and is reflected. If the DRs of the normal to the plane are 1, –1, 1, then $d.c$'s of the reflected ray are

(A) $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$ (B) $-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$

(C) $-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}$ (D) $-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$

21. A variable plane moves so that the sum of reciprocals of its intercepts on the three coordinate axes is constant λ. It passes through a fixed point, which has coordinates

(A) $(\lambda, \lambda, \lambda)$ (B) $\left(\frac{1}{\lambda}, \frac{1}{\lambda}, \frac{1}{\lambda}\right)$

(C) $(-\lambda, -\lambda, -\lambda)$ (D) $\left(-\frac{1}{\lambda}, -\frac{1}{\lambda}, -\frac{1}{\lambda}\right)$

22. Equation of the sphere with centre in the positive octant which passess through the circle $x^2 + y^2 = 4$, $z = 0$ and is cut by the plane $x + 2y + 2z = 0$ in a circle of radius 3 is

(A) $x^2 + y^2 + z^2 - 6x - 4 = 0$
(B) $x^2 + y^2 + z^2 - 6z + 4 = 0$
(C) $x^2 + y^2 + z^2 - 6z - 4 = 0$
(D) $x^2 + y^2 + z^2 - 6y - 4 = 0$

23. The equation of the sphere touching the three coordinate planes is

(A) $\Sigma x^2 + 2a(x + y + z) + 2a^2 = 0$
(B) $\Sigma x^2 - 2a(x + y + z) + 2a^2 = 0$
(C) $\Sigma x^2 \pm 2a(x + y + z) + 2a^2 = 0$
(D) $\Sigma x^2 \pm 2ax \pm 2ay \pm 2az + 2a^2 = 0$

24. The line $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ touches the sphere $\mathbf{r}^2 - 2\mathbf{r}\cdot\mathbf{c} + \mathbf{h} = 0$, $c^2 > h$ at the point with position vector a if

(A) $(\mathbf{a} - \mathbf{b})\cdot\mathbf{c} = 0$
(B) $(\mathbf{a} - \mathbf{c})\cdot\mathbf{b} = 0$
(C) $(\mathbf{b} - \mathbf{c})\cdot\mathbf{a} = 0$
(D) $\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} = 0$

25. Equation of the projection of the line $8x - y - 7z = 8$, $x + y + z = 1$ on the plane $5x - 4y - z = 5$ is

(A) $\frac{x-1}{1} = \frac{y}{2} = \frac{z}{-3}$ (B) $\frac{x}{1} = \frac{y-1}{2} = \frac{z}{-3}$

(C) $\frac{x}{1} = \frac{y}{2} = \frac{z-1}{-3}$ (D) $\frac{x}{1} = \frac{y+1}{-2} = \frac{z+1}{3}$

26. The cartesian equation of the plane
$\mathbf{r} = (1 + \lambda - \mu)\mathbf{i} + (2 + \lambda)\mathbf{j} + (3 - 2\lambda + 2\mu)\mathbf{k}$ is

(A) $2x + y = 5$ (B) $2x - y = 5$

(C) $2x + z = 5$ (D) $2x - z = 5$

27. The angle between the straight lines whose direction cosines are given by $2l + 2m - n = 0$, $mn + nl + lm = 0$, is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$ (D) none of these

28. If a variable line in two adjacent positions has direction cosines l, m, n and $l + \delta l, m + \delta m, n + \delta n$, then the small angle $\delta\theta$ between the two positions is given by

(A) $\delta\theta^2 = 4(\delta l^2 + \delta m^2 + \delta n^2)$

(B) $\delta\theta^2 = 2(\delta l^2 + \delta m^2 + \delta n^2)$

(C) $\delta\theta^2 = (\delta l^2 + \delta m^2 + \delta n^2)$

(D) none of these

29. If l_1, m_1, n_1 and l_2, m_2, n_2 are d.c.'s of the two lines inclined to each other at an angle θ, then the d.c.'s of the internal bisector of the angle between these lines are

(A) $\frac{l_1 + l_2}{2\sin\theta/2}, \frac{m_1 + m_2}{2\sin\theta/2}, \frac{n_1 + n_2}{2\sin\theta/2}$

(B) $\frac{l_1 + l_2}{2\cos\theta/2}, \frac{m_1 + m_2}{2\cos\theta/2}, \frac{n_1 + n_2}{2\cos\theta/2}$

(C) $\frac{l_1 - l_2}{2\sin\theta/2}, \frac{m_1 - m_2}{2\sin\theta/2}, \frac{n_1 - n_2}{2\sin\theta/2}$

(D) $\frac{l_1 - l_2}{2\cos\theta/2}, \frac{m_1 - m_2}{2\cos\theta/2}, \frac{n_1 - n_2}{2\cos\theta/2}$

30. The plane $lx + my = 0$ is rotated about its line of intersection with the plane $z = 0$ through an angle α. The equation of the plane in its new position is

(A) $lx + my \pm z\sqrt{l^2 + m^2}\sin\alpha = 0$

(B) $lx + my \pm z\sqrt{l^2 + m^2}\tan\alpha = 0$

(C) $lx + my \pm z\sqrt{l^2 + m^2}\cot\alpha = 0$

(D) none of these

31. P is any point on the plane $lx + my + nz = p$; a point Q is taken on the line OP such that $OP \cdot OQ = p^2$, then the locus of Q is

(A) $lx + my + nz = p(x^2 + y^2 + z^2)$

(B) $p(lx + my + nz) = x^2 + y^2 + z^2$

(C) $p(x + y + z) = lx^2 + my^2 + nz^2$

(D) none of these

32. The planes $3x - y + z + 1 = 0$, $5x + y + 3z = 0$ intersect in the line PQ. The equation of the plane through the point (2, 1, 4) and perpendicular to PQ is

(A) $x + y - 2z = 5$ (B) $x + y - 2z = -5$

(C) $x + y + 2z = 5$ (D) $x + y + 2z = -5$.

33. The equation of the plane containing the lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}$ is

(A) $\mathbf{r} \cdot (\mathbf{a}_1 - \mathbf{a}_2) \times \mathbf{b} = [\mathbf{a}_1\mathbf{a}_2\mathbf{b}]$

(B) $\mathbf{r} \cdot (\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b} = [\mathbf{a}_1\mathbf{a}_2\mathbf{b}]$

(C) $\mathbf{r} \cdot (\mathbf{a}_1 + \mathbf{a}_2) \times \mathbf{b} = [\mathbf{a}_2\mathbf{a}_1\mathbf{b}]$

(D) none of these

34. The equation of the sphere inscribed in a tetrahedron, whose faces are $x = 0$, $y = 0$, $z = 0$ and $x + 2y + 2z = 1$ is

(A) $32(x^2 + y^2 + z^2) + 8(x + y + z) + 1 = 0$

(B) $32(x^2 + y^2 + z^2) - 8(x + y + z) - 1 = 0$

(C) $32(x^2 + y^2 + z^2) - 8(x + y + z) + 1 = 0$

(D) none of these

35. The perpendicular distance of a corner of a unit cube from a diagonal not passing through it is

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{2}{\sqrt{3}}$

(C) $\sqrt{\frac{2}{3}}$ (D) none of these

36. The centre of the sphere which touches the lines $y = x$, $z = c$ and $y = -x$, $z = -c$ lies on

(A) $xy = 2cz$ (B) $xy = -2cz$

(C) $yz = 2cx$ (D) $yz = -2cx$

37. If P be any point on the plane $lx + \text{my} + nz = p$ and Q be a point on the line OP such that $OP.\ OQ = p^2$. The locus of the point Q is

(A) $lx + my + nz = x^2 + y^2 + z^2$

(B) $lx + my + nz = p(x^2 + y^2 + z^2)$

(C) $p(lx + my + nz) = x^2 + y^2 + z^2$

(D) none of these

38. Through a point $P(h, k, l)$ a plane is drawn at right angles to OP to meet the coordinate axes in A, B and C. If $OP = p$, then the area of $\Delta\, ABC$ is

(A) $\frac{p^5}{2hkl}$ (B) $\frac{p^5}{hkl}$

(C) $\frac{p^5}{4hkl}$ (D) none of these

More than One Option Correct Type

39. A variable plane passes through a fixed point (a, b, c) and meets the coordinate axes in A, B, C. The locus of the point common to the planes through A, B, C parallel to coordinate planes is

(A) $ayz + bzx + cxy = xyz$
(B) $ayz + bzx + cxy = 2xyz$
(C) $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$
(D) none of these

40. If $OABC$ is a tetrahedron such that $OA^2 + BC^2 = OB^2 + CA^2 = OC^2 + AB^2$, then

(A) AB is perpendicular to OC
(B) BC is perpendicular to OA
(C) CA is perpendicular to OB
(D) AB is perpendicular to CA

41. If the median through A of a ΔABC having vertices $A \equiv (2, 3, 5)$, $B \equiv (-1, 3, 2)$ and $C \equiv (\lambda, 5, \mu)$ is equally inclined to the axes, then

(A) $\lambda = 7$ (B) $\mu = 10$
(C) $\lambda = 10$ (D) $\mu = 7$

Match the Column Type

42.

Column-I	Column-II
I. The centre of the sphere having the circle $x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 = 0$, $5x - 2y + 4z + 7 = 0$ as the great circle is	(A) $(-1, 4, -2)$
II. The plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ at the point	(B) $(2, -3, 1)$
III. $A(3, 2, 0)$, $B(5, 3, 2)$, $C(-9, 6, -3)$ are three points forming a triangle. If AD, the bisector of $\angle BAC$ meets BC in D, then coordinates of D are	(C) $(-1, -1, -1)$
IV. The point in which the line $\frac{x+1}{-1} = \frac{y-12}{5} = \frac{z-7}{-2}$ cuts the surface $11x^2 - 5y^2 - z^2 = 0$ is	(D) $\left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$

43.

Column-I	Column-II
I. The radius of the circle $x^2 + y^2 + z^2 = 49$, $2x + 3y - z - 5\sqrt{14} = 0$ is	(A) $\frac{10}{3\sqrt{3}}$
II. The locus of the foot of the perpendicular from the origin on the variable plane through the fixed point $(2, -4, 6)$ is a sphere of radius	(B) $\sqrt{14}$
III. The distance of the line L whose vector equation is $\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 4\mathbf{k})$ from the plane π whose vector equation is $\mathbf{r} \cdot (\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = 5$, is	(C) 3
IV. The equation of the plane which meets the axes in A, B and C, given that the centroid of the triangle ABC is the point (α, β, γ), is $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = k$, where $k =$	(D) $2\sqrt{6}$

Previous Year's Questions

44. A plane which passes through die point $(3, 2, 0)$ and the line $\frac{x-4}{1} = \frac{y-7}{5} = \frac{z-4}{4}$ is: **[2002]**

(A) $x - y + z = 1$ (B) $x + y + z = 5$
(C) $x + 2y - z = 1$ (D) $2x - y + z = 5$

45. A parallelopiped is formed by planes drawn through the points $(2, 3, 5)$ and $(5, 9, 7)$, parallel to the coordinate planes. The length of a diagonal of the parallelopiped is: **[2002]**

(A) 7 unit (B) $\sqrt{38}$ unit
(C) $\sqrt{155}$ unit (D) none of these

46. The equation of the plane containing the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where: **[2002]**

(A) $ax_1 + by_1 + cz_1 = 0$ (B) $al + bm + cn = 0$

(C) $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$ (D) $lx_1 + my_1 + nz_1 = 0$

47. A tetrahedron has vertices at $O(0, 0, 0)$, $A(1, 2, 1)$, $B(2, 1, 3)$ and $C(-1, 1, 2)$. Then the angle between the faces OAB and ABC will be **[2003]**

(A) $\cos^{-1}\left(\frac{19}{35}\right)$ (B) $\cos^{-1}\left(\frac{17}{31}\right)$

(C) 30° (D) 90°

48. A line makes the same angle θ, with each of the x and z axis. If the angle β, which it makes with y-axis, is such that $\sin^2\beta = 3\sin^2\theta$, then $\cos^2\theta$ equals **[2004]**

(A) $\frac{2}{3}$ (B) $\frac{1}{5}$

(C) $\frac{3}{5}$ (D) $\frac{2}{5}$

49. Distance between two parallel planes $2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is **[2004]**

(A) $\frac{3}{2}$ (B) $\frac{5}{2}$

(C) $\frac{7}{2}$ (D) $\frac{9}{2}$

50. A line with direction cosines proportional to 2, 1, 2 meets each of the lines $x = y + a = z$ and $x + a = 2y = 2z$. The co-ordinates of each of the point of intersection are given by **[2004]**

(A) $(3a, 3a, 3a)$, (a, a, a)
(B) $(3a, 2a, 3a)$, (a, a, a)
(C) $(3a, 2a, 3a)$, $(a, a, 2a)$
(D) $(2a, 3a, 3a)$, $(2a, a, a)$

51. If the straight lines $x = 1 + s$, $y = -3 - \lambda s$, $z = 1 + \lambda s$ and $x = \frac{t}{2}, y = 1 + t, z = 2 - t$ with parameters s and t respectively, are co-planar then λ. Equals **[2004]**

(A) –2 (B) –1

(C) $-\frac{1}{2}$ (D) 0

52. The intersection of the spheres $x^2 + y^2 + z^2 + 7x - 2y - z = 13$ and $x^2 + y^2 + z^2 - 3x + 3y + 4z = 8$ is the same as the intersection of one of the sphere and the plane **[2004]**

(A) $x - y - z = 1$ (B) $x - 2y - z = 1$
(C) $x - -2z = 1$ (D) $2x - y - z = 1$

53. If the angle Q between the line $\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-2}{2}$ and the plane $2x - y + \sqrt{\lambda z} + 4 = 0$ is such that $\sin\theta = \frac{1}{3}$ the value of λ is **[2005]**

(A) $\frac{5}{3}$ (B) $\frac{-3}{5}$

(C) $\frac{3}{4}$ (D) $\frac{-4}{3}$

54. If the plane $2ax - 3ay + 4az + 6 = 0$ passes through the midpoint of the line joining the centres of the spheres **[2005]**

$x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$ and
$x^2 + y^2 + z^2 - 10x + 4y - 2z = 8$, then a equals

(A) –1 (B) 1
(C) –2 (D) 2

55. The two lines $x = ay + b$, $z = cy + d$; and $x = a'y + b'$, $z = c'y + d'$ are perpendicular to each other if **[2006]**

(A) $aa' + cc' = -1$
(B) $aa' + cc' = 1$
(C) $\frac{a}{a'} + \frac{c}{c'} = -1$

56. The image of the point (–l, 3, 4) in the plane $x - 2y = 0$ is **[2006]**

(A) $\left(-\frac{17}{3}, -\frac{19}{3}, 4\right)$ (B) (15, 11, 4)

(C) $\left(-\frac{17}{3}, -\frac{19}{3}, 1\right)$ (D) (8, 4, 4)

57. Let L be the line of intersection of the planes $2x + 3y + z = 1$ and $x + 3y + 2z = 2$. If L makes an angles α with the positive x-axis, then $\cos\alpha$ equals **[2007]**

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{1}{2}$

(C) 1 (D) $\frac{1}{\sqrt{2}}$

58. If a line makes an angle of $\frac{\pi}{4}$ with the positive directions of each of x-axis and y-axis, then the angle that the line makes with the positive direction of the z-axis is **[2007]**

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

59. If (2, 3, 5) is one end of a diameter of the sphere $x^2 + y^2 + z^2 - 6x - 12y - 2z + 20 = 0$, then the coordinates of the other end of the diameter are **[2007]**

(A) (4, 9, –3) (B) (4, –3, 3)
(C) (4, 3, 5) (D) (4, 3, –3)

60. The line passing through the points $(5, 1, a)$ and $(3, b, 1)$ crosses the yz-plane at the point $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$ then **[2008]**

(A) $a = 2, b = 8$
(B) $a = 4, b = 6$
(C) $a = 6, b = 4$
(D) $a = 8, b = 2$

61. If the straight lines $\frac{x-1}{k} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x-2}{3} = \frac{y-3}{k} = \frac{z-1}{2}$ intersect at a point, then the integer k is equal to **[2008]**

(A) −5
(B) 5
(C) 2
(D) −2

62. Let the line $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ lies in the plane $x + 3y - \alpha z + \beta = 0$. Then (α, β) equals **[2009]**

(A) (6, −17)
(B) (−6, 7)
(C) (5, −15)
(D) (−5, 15)

63. A line AB in 3-dimensional space makes angles 45° and 120° with the positive x-axis and the positive y-axis respectively. If AB makes an acute angle θ with the positive z-axis, then θ equals **[2010]**

(A) 45°
(B) 60°
(C) 75°
(D) 30°

64. If the angle between the line $x = \frac{y-1}{2} = \frac{z-3}{\lambda}$ and the plane $x + 2y + 3z =$ is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then λ equals **[2011]**

(A) $\frac{3}{2}$
(B) $\frac{2}{5}$
(C) $\frac{5}{3}$
(D) $\frac{2}{3}$

65. Statement - 1: The point $A(1, 0, 7)$ is the mirror image of the point $B(1, 6, 3)$ in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$. **[2011]**

Statement - 2: The line: $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ bisects the line segment joining $A(1, 0, 7)$ and $B(1, 6, 3)$.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement - 1
(B) Statement-1 is true, Statement-2 is false.
(C) Statement-1 is false, Statement-2 is true.
(D) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

66. An equation of a plane parallel to the plane $x - 2y + 2z = 5$ and at a unit distance from the origin is **[2012]**

(A) $x - 2y + 2z - 3 = 0$
(B) $x - 2y + 2z + 1 = 0$
(C) $x - 2y + 2z - 1 = 0$
(D) $x - 2y + 2z + 5 = 0$

67. If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then the value of k is equal to **[2012]**

(A) −1
(B) $\frac{2}{9}$
(C) $\frac{9}{2}$
(D) 0

68. If the lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar, then k can have **[2013]**

(A) exactly one value
(B) exactly two values
(C) exactly three values
(D) any value

69. Distance between two parallel planes $2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is **[2013]**

(A) $\frac{5}{2}$
(B) $\frac{7}{2}$
(C) $\frac{9}{2}$
(D) $\frac{3}{2}$

70. The image of the line $\frac{x-1}{3} = \frac{y-3}{1} = \frac{z-4}{-5}$ on the plane $2x - y + z + 3 = 0$ is the line **[2014]**

(A) $\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$
(B) $\frac{x+3}{-3} = \frac{y-5}{-1} = \frac{z+2}{5}$
(C) $\frac{x-3}{3} = \frac{y+5}{1} = \frac{z-2}{-5}$
(D) $\frac{x-3}{-3} = \frac{y+5}{-1} = \frac{z-2}{5}$

71. The angle between the two lines whose direction cosines satisfy the equations $l + m + n = 0$ and $l^2 = m^2 + n^2$ is **[2014]**

(A) $\frac{\pi}{3}$
(B) $\frac{\pi}{4}$
(C) $\frac{\pi}{6}$
(D) $\frac{\pi}{2}$

72. The distance of the point (1, 0, 2) from the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 16$, is: **[2015]**

(A) 8
(B) $3\sqrt{21}$
(C) 13
(D) $2\sqrt{14}$

73. The equation of the plane containing the lines $2x - 5y + z = 3$ and $x + y + 4z = 5$, and parallel to the plane, $x + 3y + 6z = 1$, is: **[2015]**

(A) $x + 3y + 6z = -7$
(B) $x + 3y + 6z = 7$
(C) $2x + 6y + 12z = -13$
(D) $2x + 6y + 12z = 13$

74. The distance of the point $(1, -5, 9)$ from the plane $x - y + z = 5$ measured along the line $x = y = z$ is: **[2016]**

(A) $\frac{20}{3}$ (B) $3\sqrt{10}$
(C) $10\sqrt{3}$ (D) $\frac{10}{\sqrt{3}}$

75. If the line $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$ lies in the plane, $lx + my - z = 9$, then $l^2 + m^2$ is equal to: **[2016]**

(A) 2 (B) 26
(C) 18 (D) 5

ANSWER KEYS

Single Option Correct Type

1. (C) **2.** (D) **3.** (B) **4.** (A) **5.** (A) **6.** (A) **7.** (A) **8.** (B) **9.** (A) **10.** (C)
11. (D) **12.** (A) **13.** (B) **14.** (C) **15.** (D) **16.** (B) **17.** (D) **18.** (B) **19.** (B) **20.** (D)
21. (B) **22.** (C) **23.** (D) **24.** (B) **25.** (A) **26.** (C) **27.** (A) **28.** (C) **29.** (B) **30.** (B)
31. (B) **32.** (B) **33.** (B) **34.** (C) **35.** (C) **36.** (B) **37.** (C) **38.** (A)

More than One Option Correct Type

39. (A, C) **40.** (A, B, C) **41.** (A, B)

Match the Column Type

42. I ↔ (C), II ↔ (A), III ↔ (D), IV ↔ (B) **43.** I ↔ (D), II ↔ (B), III ↔ (A), IV ↔ (C)

Previous Year's Questions

44. (A) **45.** (A) **46.** (C) **47.** (C) **48.** (C) **49.** (C) **50.** (B) **51.** (A) **52.** (D) **53.** (A)
54. (C) **55.** (A) **56.** (C) **57.** (A) **58.** (D) **59.** (A) **60.** (C) **61.** (A) **62.** (B) **63.** (B)
64. (D) **65.** (D) **66.** (A) **67.** (C) **68.** (B) **69.** (B) **70.** (A) **71.** (A) **72.** (C) **73.** (B)
74. (C) **75.** (A)

HINTS AND SOLUTIONS

Single Option Correct Type

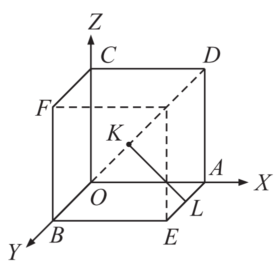
1. Any plane parallel to x-axes is $by + cz + d = 0$.
If it passes through (2, 3, 1) and (4, –5, 3), then
$3b + c + d = 0$ and $-5b + 3c + d = 0$,

i.e., $\frac{b}{1-3} = \frac{c}{-5-3} = \frac{d}{9+5}$, i.e., $\frac{b}{1} = \frac{c}{4} = \frac{d}{-7}$.

Hence, the plane parallel to x-axis is $\mathbf{y + 4z - 7 = 0}$.

2. Requaried distance = KL

$$= \sqrt{\left(a - \frac{a}{2}\right)^2 + 0^2 + \left(0 - \frac{a}{2}\right)^2} = \frac{a}{\sqrt{2}}.$$



3. When folded co-ordinates will be $D(0, 0, a)$; $C(a, 0, 0)$; $A(-a, 0, 0)$; $B(0, -a, 0)$

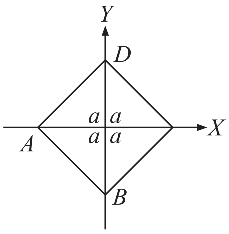
Equation of DC is,

$$\frac{x}{a} = \frac{y}{0} = \frac{z-a}{-a}$$

Equation of AB is,

$$\frac{x+a}{a} = \frac{y}{-a} = \frac{z}{0}$$

$\therefore$ Shortest distance $\frac{2a}{\sqrt{3}}$.



4. The line of intersection of the planes
$\mathbf{r} \cdot (3\mathbf{i} - \mathbf{j} + \mathbf{k}) = 1$ and
$\mathbf{r} \cdot (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 2$ is $\perp$ to each of the normal vectors
$\mathbf{n}_1 = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$
$\therefore$ It is parallel to the vector
$\mathbf{n}_1 \times \mathbf{n}_2 = (3\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$
$= \mathbf{-2i + 7j + 13k}$.

5. Let the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$
It passes through $(1, 0, 0) \Rightarrow 1 + 2u + d = 0$ (1)
It passes through $(0, 1, 0) \Rightarrow 1 + 2v + d = 0$ (2)
It passes through $(0, 0, 1) \Rightarrow 1 + 2w + d = 0$ (3)

$$\therefore \quad u = v = w = -\frac{1+d}{2}$$

Radius of the sphere $= \sqrt{u^2 + v^2 + w^2 - d}$

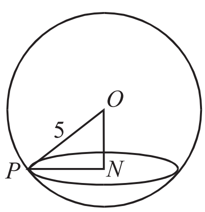
$$= \frac{1}{2}\sqrt{3(1+d)^2 - 4d}$$

$$= \frac{1}{2}\sqrt{3d^2 + 2d + 3}$$

Now, $3d^2 + 2d + 3 \geq \frac{4 \times 3 \times 3 - (2)^2}{4 \times 3} = \frac{32}{12} = \frac{8}{3}$

$$\therefore \quad \text{radius} \geq \frac{1}{2} \times \sqrt{\frac{8}{3}} = \sqrt{\frac{2}{3}}$$

6. The equation of ON is $\mathbf{r} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$ (1)



Since it passes through the origin and is parallel to the vector $(\mathbf{i} + \mathbf{j} + \mathbf{k})$, any pt. on it is $\lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$. If this pt. lies on the plane $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 3\sqrt{3}$
then, $\lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 3\sqrt{3}$
or, $\lambda(1 + 1 + 1) = 3\sqrt{3}$
$\therefore \quad \lambda = \sqrt{3}$
Putting the value of λ in (1), we get the position vector N i.e., centre of the circle as $\sqrt{3}\,(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

7. Distance of (α, β, γ) from y-axis is given by
$d = \sqrt{\alpha^2 + \gamma^2}$
$\therefore$ Distance (d) of $(3, 4, 5)$ from y-axis is
$d = \sqrt{3^2 + 5^2} = \sqrt{9 + 25} = \sqrt{\mathbf{34}}$.

8. Let the foot of the perpendicular from the origin on the given plane be $P(\alpha, \beta, \gamma)$. Since, the plane passes through $A(a, b, c)$.
$\therefore AP \perp OP \quad \Rightarrow \quad \alpha(\alpha - a) + \beta(\beta - b) + \gamma(\gamma - c) = 0$
Hence, the locus of (α, β, γ) is
$x(x - a) + y(y - b) + z(z - c) = 0$
$\Rightarrow \quad x^2 + y^2 + z^2 - ax - by - cz = 0$,
which is a sphere of radius $\mathbf{\frac{1}{2}\sqrt{a^2 + b^2 + c^2}}$

9. If l, m, n are the d.c's of the line, then
$1 \cdot l - 1 \cdot m + 1 \cdot n = 0$
and, $1 \cdot l - 3 \cdot m + 0 \cdot n = 0$

$$\therefore \quad \frac{l}{0+3} = \frac{m}{1-0} = \frac{n}{-3+1}$$

Hence, the d.r's of the line are **3, 1, –2.**

10. The straight line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ meets the plane $\mathbf{r} \cdot \mathbf{n} = 0$ in P for which λ is given by
$(\mathbf{a} + \lambda\mathbf{b}) \cdot \mathbf{n} = 0 \Rightarrow \lambda = -\frac{\mathbf{a.n}}{\mathbf{b.n}}$
Thus, the positive vector of P is

$$\mathbf{r} = \mathbf{a} - \frac{\mathbf{a.n}}{\mathbf{b.n}}\mathbf{b}$$

11. Suppose, any line l through the given point $(1, -2, 3)$ meets the sphere $x^2 + y^2 + z^2 = 4$ in the point (x, y, z).
Then, $x_1^2 + y_1^2 + z_1^2 = 4$ (1)
Now, let the coordinates of the point which divides the join of $(1, -2, 3)$ and (x_1, y_1, z_1) in the ratio $2 : 3$ be (x_2, y_2, z_2). Then, we have

$$x_2 = \frac{2.x_1 3.1}{2+3} \quad \text{or} \quad x_1 = \frac{5x-3}{2}$$

$$y_2 = \frac{2.y_1 + 3(-2)}{2+3} \quad \text{or} \quad y_1 = \frac{5y_2 + 6}{2} \qquad (2)$$

$$z_2 = \frac{2.z_1 + 3.3}{2+2} \quad \text{or} \quad z_1 = \frac{5z_2 - 9}{2}$$

Putting the values of x_1, y_1, z_1 from (2) in (1), we have
$(5x_2 - 3)^2 + (5y_2 + 6)^2 + (5z_2 - 9)^2 = 4 \times 4$
$\Rightarrow \quad 25(x_2^2 + y_2^2 + z_2^2) - 30x_2 + 60y_2 - 90z_2 + 110 = 0$

$\therefore$ The locus of (x_2, y_2, z_2) is

$5(x^2 + y^2 + z^2) - 6(x - 2y + 3z) + 22 = 0$

12. The vector equation of the plane passing through points **a**, **b**, **c** is

$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) = [\mathbf{a\ b\ c}]$.

Therefore, the length of the ⊥ from the origin to this plane is given by

$$\frac{[\mathbf{a\,b\,c}]}{|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|}.$$

13. The lines $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} \times \mathbf{c})$ and $\mathbf{r} = \mathbf{b} + \mu(\mathbf{c} \times \mathbf{a})$ pass through points **a** and **b**, respectively and are parallel to vector $\mathbf{b} \times \mathbf{c}$ and $\mathbf{c} \times \mathbf{a}$, respectively. Therefore, they intersect if $\mathbf{a} - \mathbf{b}$, $\mathbf{b} \times \mathbf{c}$ and $\mathbf{c} \times \mathbf{a}$ are coplanar and so

$(\mathbf{a} - \mathbf{b}) \cdot \{(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})\} = 0$

$\Rightarrow$ $(\mathbf{a} - \mathbf{b}) \cdot ([\mathbf{b\ c\ a}]\mathbf{c} - [\mathbf{b\ c\ c}]\mathbf{a}) = 0$

$\Rightarrow$ $((\mathbf{a} - \mathbf{b}) \cdot \mathbf{c})\,[\,\mathbf{b\ c\ a}] = 0$

$\Rightarrow$ $\mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c} = 0 \quad \Rightarrow \quad \mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}$.

14. The given plane passes through **a** and parallel to the vectors $\mathbf{b} - \mathbf{a}$ and **c**.

$\therefore$ it is normal to $(\mathbf{b} - \mathbf{a}) \times \mathbf{c}$.

Hence the equation is $(\mathbf{r} - \mathbf{a}) \cdot [(\mathbf{b} - \mathbf{a}) \times \mathbf{c}] = 0$

or, $\mathbf{r} \cdot [\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}] = [\mathbf{a\ b\ c}]$

$\therefore$ The length of the ⊥ from the origin to this plane is

$$\frac{[\mathbf{a\,b\,c}]}{|\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|}.$$

15. Any plane through the intersection of $\mathbf{r} \cdot \mathbf{a} = p$

and, $\mathbf{r} \cdot \mathbf{b} = q$ is

$\mathbf{r} \cdot (\mathbf{a} - \lambda\mathbf{b}) = p - \lambda q$ (1)

Since it passes through the origin

$\therefore$ $\mathbf{0} \cdot (\mathbf{a} - \lambda\mathbf{b}) = p - \lambda q$

$\Rightarrow$ $p - \lambda q = 0 \quad \Rightarrow \quad \lambda = \dfrac{p}{q}$.

Putting this value of λ in (1), we get

$\mathbf{r} \cdot \left(\mathbf{a} - \dfrac{p}{q}\mathbf{b}\right) = p - \dfrac{p}{q}q = 0$

i.e., $\mathbf{r} \cdot (\mathbf{a}q - p\mathbf{b}) = 0$

This is the required equation.

16. The line of intersection of the planes $\mathbf{r} \,.\, (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 0$ and $\mathbf{r}.(3\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 0$ is parallel to the vector $(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times (3\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = -4\mathbf{k} + 8\mathbf{j} - 4\mathbf{k}$. Since both the planes pass through the origin, therefore their line of intersection will also pass through the origin. Thus, the required line passes through the origin and is parallel to the vector $-4\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$. Hence, its equation is

$\mathbf{r} = \mathbf{0} + \lambda'(-4\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}) \Rightarrow \mathbf{r} = \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ where $\lambda = -4\lambda'$.

17. Distance of plane $x + y + z = 5\sqrt{3}$ from centre (0, 0, 0) of sphere $x^2 + y^2 + z^2 = 5$

$= \dfrac{5\sqrt{3}}{\sqrt{1^2 + 1^2 + 1^2}} = 5$ (radius of sphere), so plane touches it.

18. Since **OP** has projections $\dfrac{13}{5}, \dfrac{19}{5}$ and $\dfrac{26}{5}$ on the coordinate axes, therefore $\mathbf{OP} = \dfrac{13}{5}\mathbf{i} + \dfrac{19}{5}\mathbf{j} + \dfrac{26}{5}\mathbf{k}$. Suppose P divides the join of $Q(2, 2, 4)$ and $R(3, 5, 6)$ in the ratio $\lambda : 1$. Then, the position vector of P is

$\left(\dfrac{3\lambda + 2}{\lambda + 1}\right)\mathbf{i} + \left(\dfrac{5\lambda + 2}{\lambda + 1}\right)\mathbf{j} + \left(\dfrac{6\lambda + 4}{\lambda + 1}\right)\mathbf{k}.$

$\therefore$ $\dfrac{13}{5}\mathbf{i} + \dfrac{19}{5}\mathbf{j} + \dfrac{26}{5}\mathbf{k}$

$= \left(\dfrac{3\lambda + 2}{\lambda + 1}\right)\mathbf{i} + \left(\dfrac{5\lambda + 2}{\lambda + 1}\right)\mathbf{j} + \left(\dfrac{6\lambda + 4}{\lambda + 1}\right)\mathbf{k}.$

$\Rightarrow$ $\dfrac{3\lambda + 2}{\lambda + 1} = \dfrac{13}{5}, \dfrac{5\lambda + 2}{\lambda +} = \dfrac{19}{5}, \dfrac{6\lambda + 4}{\lambda + 1} = \dfrac{26}{5}$

$\Rightarrow$ $2\lambda = 3 \Rightarrow \lambda = 3/2$.

19. *PA, PB* are perpendiculars drawn from $P(a, b, c)$ on *yz* and *zx* planes

$\therefore$ $A(0, b, c)$ and $B(a, 0, c)$ are points on *yz* and *zx* planes.

The equation of plane passing through (0, 0, 0) is

$Ax + By + Cz = 0$

which also passes through A and B

$\therefore$ $A \cdot 0 + B \cdot b + C \cdot c = 0$ (1)

$A \cdot a + B \cdot 0 + C \cdot c = 0$ (2)

Solving (1) and (2), we get

$\Rightarrow$ $\dfrac{A}{bc - 0} = \dfrac{B}{ac - 0} = \dfrac{C}{0 - ab} = \lambda$

$\Rightarrow$ $A = \lambda bc, B = \lambda ac, C = -\lambda ab$

Required equation is

$bcx + acy - abz = 0$

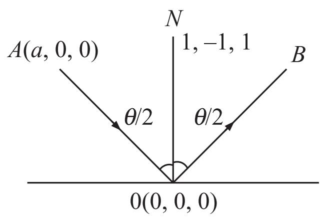
20. Let the source of light be situated at $A(a, 0, 0)$, where $a \neq 0$. Let OA be the incident ray and OB the reflected ray. ON is the normal to the mirror at O.

$\therefore$ $\angle AON = \angle NOB = \dfrac{\theta}{2}$ (say)

d.r's of OA are a, 0, 0 and so its *d.r*'s are 1, 0, 0

d.r's of ON are $\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$

$\therefore$ $\cos\dfrac{\theta}{2} = \dfrac{1}{\sqrt{3}}$



Let l, m, n be the *d.r*'s of the reflected ray OB. Then,

$\dfrac{l + 1}{2\cos\theta/2} = \dfrac{1}{\sqrt{3}}, \dfrac{m + 0}{2\cos\theta/2} = -\dfrac{1}{\sqrt{3}}$

and, $\dfrac{n+0}{2\cos\theta/2} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow\ l = \dfrac{2}{3} - 1,\ m = \dfrac{-2}{3},\ n = \dfrac{2}{3}$

$\Rightarrow\ l = -\dfrac{1}{3},\ m = -\dfrac{2}{3},\ n = \dfrac{2}{3}.$

Hence, *d.c*'s of the reflected ray are $-\dfrac{1}{3}, -\dfrac{2}{3}, \dfrac{2}{3}$.

21. Let equation of the variable plane be

$\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ (1)

The intercepts on the coordinate axes are a, b, c. The sum of reciprocals of intercepts is constant λ, therefore

$\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = \lambda$ or $= \dfrac{(1/\lambda)}{a} + \dfrac{(1/\lambda)}{b} + \dfrac{(1/\lambda)}{c} = \lambda$

$\therefore\ \left(\dfrac{1}{\lambda}, \dfrac{1}{\lambda}, \dfrac{1}{\lambda}\right)$ lies on the plane (1)

Hence, the variable plane (1) always passes through the fixed point $\left(\dfrac{1}{\lambda}, \dfrac{1}{\lambda}, \dfrac{1}{\lambda}\right)$.

22. Equation of a sphere through the given circle is

$x^2 + y^2 + z^2 - 4\lambda + z = 0$

Centre of the sphere is $(0, 0, -\lambda/2)$ and the radius

$r = \sqrt{0 + 0 + \lambda^2/4 + 4}$

Let d the distance of the plane $x + 2y + 2z = 0$ from the centre of the sphere.

then, $d = \left|\dfrac{0 + 2\times 0 + 2(-\lambda/2)}{\sqrt{1 + 2^2 + 2^2}}\right| = \left|\dfrac{\lambda}{3}\right|.$

Since the sphere (1) cuts the plane in a circle of radius 3,

$r^2 - d^2 = 3^2 \Rightarrow \lambda^2/4 + 4 - \lambda^2/9 = 9 \Rightarrow 5\lambda^2/36 = 5$

$\Rightarrow\ \lambda^2 = 36 \Rightarrow \lambda = \pm 6$

Since the centre lies in the positive octant, $\lambda < 0$ and hence the required equation is

$x^2 + y^2 + z^2 - 6z - 4 = 0.$

23. Let the radius of the sphere be a; then the distance of its centre from coordinate planes which it is touching should be equal to radius a. Hence, its centre is (a, a, a). But since the centre can be in any octant we say that its centre is $(\pm a, \pm a, \pm a)$ and radius a, so that its equation is $(x \pm a)^2 + (y \pm a)^2 + (z \pm a)^2 = a^2$ or $x^2 + y^2 + z^2 \pm 2ax \pm 2ay \pm 2az + 2a^2 = 0$

There can be an infinite number of such spheres depending on the value of a.

In case the radius, i.e., a be fixed, then only eight such spheres can be drawn.

24. The given line meets the given sphere at points for which $(\mathbf{a} + t\mathbf{b}^2)^2 - 2(\mathbf{a} + t\mathbf{b}).\,c + h = 0$ or $t^2\,\mathbf{b}^2 + 2t\,(\mathbf{a}.\mathbf{b} - \mathbf{b}.\mathbf{c}) + \mathbf{a}^2 - 2\mathbf{a}.\mathbf{c} + h = 0$

The line $r = \mathbf{a} + t\mathbf{b}$ touches the given sphere at $r = \mathbf{a}$, if the two values of t obtained from (1) coincide with $t = 0$. $\Rightarrow \mathbf{a}^2 - 2\mathbf{a}.\mathbf{c} + h = 0$ which shows $\mathbf{a}$ lies on the sphere and $\mathbf{a}.\mathbf{b} - \mathbf{b}.\mathbf{c} = 0 \Rightarrow \mathbf{(a - c).\ b = 0.}$

25. Any plane through the given line is

$x + y + z - 1 + \lambda(8x - y - 7z - 8) = 0$ (1)

$\Rightarrow\ (1 + 8\lambda)x + (1 - \lambda)y + (1 - 7\lambda)z - 1 - 8\lambda = 0$

It is perpendicular to the plane $5x - 4y - z = 5$

if $(1 + 8\lambda)5 + (1 - \lambda)(-4) - (1 - 7\lambda) = 0 \Rightarrow \lambda = 0$

So that (1) becomes $x + y + z - 1 = 0$

Now, the line of intersection of the planes $x + y + z - 1 = 0$ and $5x - 4y - z = 5$ is the required line of projection, which clearly passes through (1, 0, 0) and if l, m, n are direction ratios of the line then $l + m + n = 0$ and $5l - 4m - n = 0$

$\Rightarrow\ \dfrac{l}{-1+4} = \dfrac{m}{5+1} = \dfrac{n}{-4-5}$

$\Rightarrow\ \dfrac{l}{3} = \dfrac{m}{6} = \dfrac{n}{-9}\ \Rightarrow\ \dfrac{l}{1} = \dfrac{m}{2} = \dfrac{n}{-3}$

and hence the requared equation of the projection is

$\dfrac{x-1}{1} = \dfrac{y}{2} = \dfrac{z}{-3}$

26. We have,

$\mathbf{r} = (1 + \lambda - \mu)\mathbf{i} + (2 - \lambda)\mathbf{j} + (3 - 2\lambda + 2\mu)\mathbf{k}$

$\Rightarrow\ \mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} - 2\mathbf{k}) + \mu(-\mathbf{i} + 2\mathbf{k}).$

which is a plane passing through

$\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and parallel to the vectors

$\mathbf{b} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + 2\mathbf{k}$.

Therefore, it is $\perp$ to the vector

$\mathbf{n} = \mathbf{b} \times \mathbf{c} = -2\mathbf{i} - \mathbf{k}$.

Hence, its vector equation is $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$

$\Rightarrow\ \mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

$\Rightarrow\ \mathbf{r} \cdot (-2\mathbf{i} - \mathbf{k}) = -2 - 3 \Rightarrow \mathbf{r} \cdot (2\mathbf{i} - \mathbf{k}) = 5$

So, the cartesian equation is

$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{k}) = 5$

or, $\mathbf{2x + z = 5}$.

27. Relations are

$2l - 2m - n = 0$ i.e., $n = 2l + 2m$ and $mn + nl + lm = 0$.

Eliminating n, we have $(m + l)(2l + 2m) + lm = 0$

i.e., $2l^2 + 5lm + 2m^2 = 0$ or $(2l + m)(l + 2m) = 0$.

When $2l + m = 0$, we have from $2l + 2m - n = 0$, $n = m$.

Thus, $\dfrac{l}{-1} = \dfrac{m}{2} = \dfrac{n}{2},$

i.e., d.c.'s of one line are proportional to [–1, 2, 2].

Again, when $l + 2m = 0$, we have from $2l + 2m - n = 0$, $n = l$.

$\therefore\ \dfrac{l}{2} = \dfrac{m}{-1} = \dfrac{n}{2},$

i.e., d.c's of other line are proportional to [2, –1, 2].

Now, if θ be the angle between the two lines, then

$$\cos\theta = \frac{-1\cdot 2 + 2\cdot -1 + 2\cdot 2}{\sqrt{(1^2 + 2^2 + 2^2)}\sqrt{(2^2 + 1^2 + 2^2)}} = 0,$$

i.e., $\theta = \dfrac{1}{2}\pi$ or the two lines are at right angles to each other.

28. $[l, m, n]$ and $[l + \delta l, m + \delta m, n + \delta n]$ are d.c.'s

$\therefore \quad l^2 + m^2 + n^2 = 1 \qquad (1)$

and, $(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$

i.e., $(l^2 + m^2 + n^2) + (\delta l^2 + \delta m^2 + \delta n^2) + 2l\delta l$

$+ 2m\delta m + 2n\delta n = 1$

or, $\delta l^2 + \delta m^2 + \delta n^2 = -2(l\delta l + m\delta m + n\delta n)$ from (1) $\qquad (2)$

Now, $\cos \delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$

$= l^2 + m^2 + n^2 + l\delta l + m\delta m + n\delta n$

$= 1 - \frac{1}{2}\, [\delta l^2 + \delta m^2 + \delta n^2]$ from (1) and (2)

or, $\delta l^2 + \delta m^2 + \delta n^2 = 2(l - \cos \delta\theta)$

$= 2 \cdot 2 \sin^2 \frac{1}{2}\, \delta\theta$

$= 4\left(\frac{1}{2}\delta\theta\right)^2$ as $\sin \frac{1}{2}\delta\theta \approx \frac{1}{2}\delta\theta$.

$= \delta\theta^2$.

29. Let OA and OB be two lines with d.c's l_1, m_1, n_1 and l_2, m_2, n_2. Let $OA = OB = 1$. Then, the coordinates of A and B are (l_1, m_1, n_1) and (l_2, m_2, n_2), respectively. Let OC be the bisector of $\angle AOB$. Then, C is the mid point of AB and so its coordinates are

$$\left(\frac{l_1 + l_2}{2}, \frac{m_1 + m_2}{2}, \frac{n_1 + n_2}{2}\right).$$

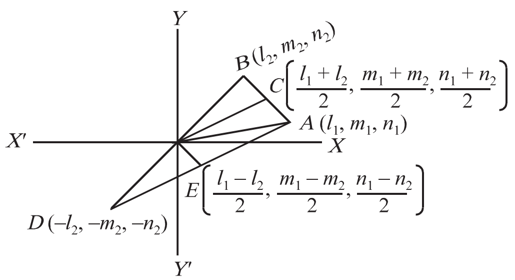
$\therefore$ d.r's of OC are $\frac{l_1 + l_2}{2}, \frac{m_1 + m_2}{2}, \frac{n_1 + n_2}{2}$.

We have, $OC = \sqrt{\left(\frac{l_1 + l_2}{2}\right)^2 + \left(\frac{m_1 + m_2}{2}\right)^2 + \left(\frac{n_1 + n_2}{2}\right)^2}$

$= \frac{1}{2}\sqrt{(l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) + 2(l_1 l_2 + m_1 m_2 + n_1 n_2)}$

$= \frac{1}{2}\sqrt{2 + 2\cos\theta} \quad [\because \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2]$

$= \frac{1}{2}\sqrt{2(1 + \cos\theta)} = \cos\left(\frac{\theta}{2}\right)$.



$\therefore$ d.c's of $\boldsymbol{OC}$ are $\frac{l_1 + l_2}{2(OC)'}, \frac{m_1 + m_2}{2(OC)'}, \frac{n_1 + n_2}{2(OC)'}$

or, $\frac{l_1 + l_2}{2\cos\theta/2}, \frac{m_1 + m_2}{2\cos\theta/2}, \frac{n_1 + n_2}{2\cos\theta/2}$.

30. The plane has been rotated about its line of intersection with the plane $z = 0$; hence it will pass through the intersection of the plane $lx + my = 0$ and $z = 0$.

Let the equation of the plane after rotation be

$$lx + my + lz = 0; \qquad (1)$$

then it makes an angle α with the plane

$$lx + my = 0. \qquad (2)$$

$\therefore \quad \cos\alpha = \frac{l \cdot l + m \cdot m + \lambda \cdot 0}{\sqrt{(l^2 + m^2 + \lambda^2)}\sqrt{(l^2 + m^2)}}$

$= \frac{\sqrt{(l^2 + m^2)}}{\sqrt{(l^2 + m^2 + \lambda^2)}}$

$\therefore \quad (l^2 + m^2 + \lambda^2)\cos^2\alpha = l^2 + m^2$

or, $\quad \lambda^2\cos^2\alpha = (l^2 + m^2)(1 - \cos^2\alpha)$,

i.e., $\quad \lambda^2 = (l^2 + m^2)\tan^2\alpha$

or, $\quad \lambda = \pm\sqrt{l^2 + m^2}\,\tan\alpha$.

Putting this value of λ in (1), the required plane is given by

$lx + my \pm z\sqrt{l^2 + m^2}\,\tan\alpha = 0$

31. Let Q be the point (x', y', z').

Also, let $OP = r$ and $OQ = r'$ $\sqrt{(x'^2 + y'^2 + z'^2)}$. Then, equation of OQ (in distance form) is

$$\frac{x}{x'/r'} = \frac{y}{y'/r'} = \frac{z}{z'/r'} = r$$

So, coordinates of the point P, which is at a distance r from O are

$$\left(r\frac{x'}{r'}, r\frac{y'}{r'}, r\frac{z'}{r'}\right)$$

i.e., $\left(\frac{x'p^2}{r'^2}, \frac{y'p^2}{r'^2}, \frac{z'p^2}{r'^2}\right)$ as $rr' = p^2$.

This point P lies on the plane $lx + my + nz = p$.

$\therefore \quad l\frac{x'p^2}{r'^2} + m\frac{y'p^2}{r'^2} + n\frac{z'p^2}{r'^2} = p,$

or, $\quad p(lx' + my' + nz') = r'^2 = (x'^2 + y'^2 + z'^2)$.

$\therefore \quad$ Locus of $Q(x', y', z')$ is

$p(lx + my + nz) = x^2 + y^2 + z^2$.

32. Let $[l, m, n]$ be the direction -cosines of PQ, then

$3l - m + n = 0$

and, $5l + m + 3n = 0$,

$\therefore \quad \frac{l}{-3-1} = \frac{m}{5-9} = \frac{n}{3+5}$ i.e., $\frac{l}{1} = \frac{m}{1} = \frac{n}{-2}$

Now, a plane $\perp$ to PQ will have l, m, n as the coefficients of x, y and z.

Hence, the plane $\perp$ to PQ is $x + y - 2z = \lambda$.

It passes through (2, 1, 4);

$\therefore \quad 2 + 1 - 2.4 = \lambda$ i.e., $\lambda = -5$.

Hence, the required plane is

$x + y - 2z = -5$.

33. The required plane passes through the points having position vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ and is parallel to the vector $\mathbf{b}$. Therefore, if $\mathbf{r}$ is the position vector of any variable point on the plane, then the vectors $\mathbf{r} - \mathbf{a}_1$, $\mathbf{a}_2 - \mathbf{a}_1$ and $\mathbf{b}$ are coplanar.

$\therefore \quad (\mathbf{r} - \mathbf{a}_1) \cdot ((\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}) = 0$

$\Rightarrow \quad \mathbf{r} \cdot (\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b} - \mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{b}) = 0$

$\Rightarrow \quad \mathbf{r} \cdot (\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b} = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{b}]$.

34. Let (a, b, c) be the centre and r, the radius of the sphere.

The sphere is inscribed in the tetrahedron, hence the length of the perpendicular from the centre (a, b, c) upon each of the faces = radius of the sphere

$$\therefore \quad \frac{a}{1} = \frac{b}{1} = \frac{c}{1} = \frac{1-a-2b-2c}{\sqrt{1+4+4}} = r$$

i.e., $a = b = c = \dfrac{1-a-2b-2c}{3} = r$ (1)

$\therefore$ From (1), we get

$3a = 1 - a - 2b - 2c$ (2)

and, $a = b = c$

$\therefore \quad 3a = 1 - a - 2a - 2a \Rightarrow 8a = 1 \;\therefore\; a = \dfrac{1}{8}$

and, then $r = a = \dfrac{1}{8}$

$\therefore$ Centre is $\left(\dfrac{1}{8}, \dfrac{1}{8}, \dfrac{1}{8}\right)$ and radius $= \dfrac{1}{8}$

Hence, the required sphere is

$$\left(x - \frac{1}{8}\right)^2 + \left(y - \frac{1}{8}\right)^2 + \left(z - \frac{1}{8}\right)^2 = \left(\frac{1}{8}\right)^2$$

$\Rightarrow \quad 32(x^2 + y^2 + z^2) - 8(x + y + z) + 1 = 0.$

35. Let the edges OA, OB, OC of the unit cube be along OX, OY, OZ, respectively. Since

$OA = OB = OC = 1$ unit

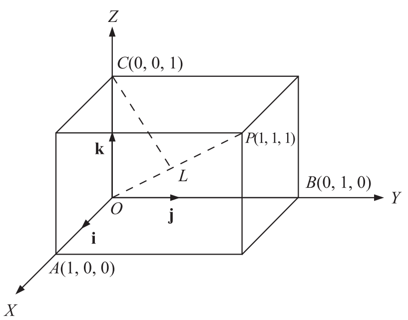
$\therefore \; \overrightarrow{OA} = \mathbf{i}, \overrightarrow{OB} = \mathbf{j}, \overrightarrow{OC} = \mathbf{k}$

Let CL be perpendicular from the corner C on the diagonal OP. The vector equation of **OP** is: $\mathbf{r} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\therefore$ **OL** = projection of **OC** on **OP** = **OC** . **OP** = **k** $\dfrac{(i+j+k)}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

Now, $OC^2 = OL^2 + CL^2 \Rightarrow CL^2 = OC^2 - OL^2 = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

$\Rightarrow \quad CL = \sqrt{\dfrac{2}{3}}$



36. Let equation of sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

The line $y = x$, $z = c$ intersects it at the point where

$2x^2 + 2(u + v)x + (c^2 + 2wc + d) = 0$

Since the line touches the sphere, the roots of this equation are coincident.

$\therefore \quad 4(u + v)^2 - 8(c^2 + 2wc + d) = 0$ (1)

Similarly, for second line

$4(u - v)^2 - 8(c^2 - 2wc + d) = 0$ (2)

Subtracting (2) from (1), we get $4uv - 8wc = 0$

$\Rightarrow uv = 2wc$

$\therefore$ Locus of centre $(-u, -v, -w)$ is $xy = -2cz$

37. Let P be the point (x_1, y_1, z_1) on the given plane, then $lx_1 + my_1 + nz_1 = p$ (1)

Let Q be (α, β, γ). Since, O, P, Q are collinear, therefore

$\dfrac{x_1}{\alpha} = \dfrac{y_1}{\beta} = \dfrac{z_1}{\gamma} = k$ (say) (2)

Now, $OP.\,OQ = p^2 \;\Rightarrow\; \sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{\alpha^2 + \beta^2 + \gamma^2} = p^2$

$\Rightarrow \quad k(\alpha^2 + \beta^2 + \gamma^2) = p^2$ (3)

Also, from (1) and (2), $k(l\alpha + m\beta + n\gamma) = p$; From (3) and (4) we have $p(l\alpha + m\beta + n\gamma) = (\alpha^2 + \beta^2 + \gamma^2)$

$\therefore$ Locus of $Q \equiv (\alpha, \beta, \gamma)$ is $p(lx + my + nz) = x^2 + y^2 + z^2$

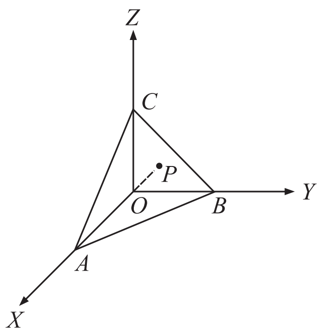
38. Here, $OP = \sqrt{h^2 + k^2 + l^2} = p$

$\therefore$ *DRs* of OP are: $\dfrac{h}{\sqrt{h^2+k^2+l^2}}, \dfrac{k}{\sqrt{h^2+k^2+l^2}}, \dfrac{l}{\sqrt{h^2+k^2+l^2}}$ or $\dfrac{h}{p}, \dfrac{k}{p}, \dfrac{l}{p}$

Since OP is normal to the plane, therefore, equation of plane is

$\dfrac{h}{p}x + \dfrac{h}{p}y + \dfrac{l}{p}z = p$ or $hx + ky + lz = p^2$

$\therefore \quad A \equiv \left(\dfrac{p^2}{h}, 0, 0\right), B \equiv \left(0, \dfrac{p^2}{k}, 0\right), C \equiv \left(0, 0, \dfrac{p^2}{l}\right)$



Now, Area of ΔABC, $\Delta = \sqrt{A_{xy}^2 + A_{yz}^2 + A_{zx}^2}$

where, A_{xy}^2 is area of projection of ΔABC on xy plane = area of ΔAOB

$$\text{Now, } A_{xy} = \frac{1}{2}\begin{vmatrix} p^2/h & 0 & 1 \\ 0 & p^2/k & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{p^4}{2|hk|}$$

Similarly, $A_{yz} = \dfrac{p^4}{2|kl|}$ and $A_{zx} = \dfrac{p^4}{2|lh|}$

$$\therefore \quad \Delta = \sqrt{A_{xy}^2 + A_{yz}^2 + A_{zx}^2} = \frac{p^5}{2hkl}$$

More than One Option Correct Type

39. Let the plane be $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$.

It passes through (a, b, c); $\therefore \frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1$. (1)

Now, coordinates of the points A, B, C are $(\alpha, 0, 0)$, $(0, \beta, 0)$ and $(0, 0, \gamma)$, respectively.

Equations of the planes through A, B, C parallel to co-ordinate planes are

$x=\alpha$ (2)

$y=\beta$ (3)

and, $z=\gamma$. (4)

The locus of their point of intersection will be obtained by eliminating α, β, γ from these with the help of the relation (1).

We thus get

$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$,

i.e., $ayz + bxz + cxy = xyz$.

which is the required locus.

40. Let $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$, $\mathbf{OC} = \mathbf{c}$, then, we have

$\mathbf{a} \cdot \mathbf{a} + (\mathbf{b} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{b} \cdot \mathbf{b} + (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a})$

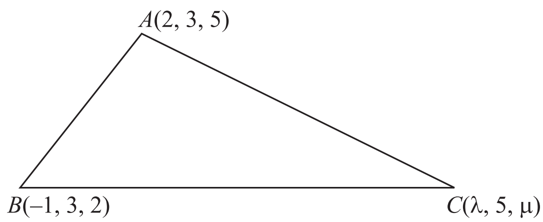
$\Rightarrow -2\mathbf{b} \cdot \mathbf{c} = -2\mathbf{c} \cdot \mathbf{a} \Rightarrow (\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} = 0$ or $\mathbf{BA} \cdot \mathbf{OC} = 0$

Therefore, AB is perpendicular to OC. Similarly, BC is perpendicular to OA and CA is perpendicular to OB.

41. Mid-point of BC is $\left(\frac{\lambda-1}{2}, 4, \frac{2+\mu}{2}\right)$

Direction ratios of median through A are

$\frac{\lambda-1}{2}-2, 4-3, \frac{2+\mu}{2}-5$, i.e., $\frac{\gamma-5}{2}, 1, \frac{\mu-8}{2}$



The median is equally inclined to axes, so, the direction ratios must be equal, so

$\frac{\lambda-5}{2}=1=\frac{\mu-8}{2} \Rightarrow \lambda=7, \mu=10$

Match the Column Type

42. I Equation of any sphere through the given circle is $x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 + \lambda(5x - 2y + 4z + 7) = 0$. Centre of the sphere is

$\left(\frac{3-5\lambda}{2}, \frac{2\lambda-4}{2}, \frac{2-4\lambda}{2}\right)$

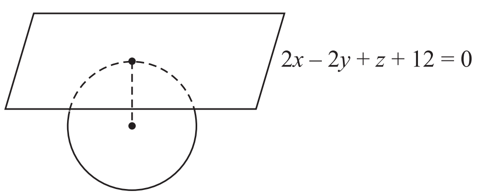
Since the given circle is a great circle, the centre of this sphere lies on the plane $5x - 2y + 4z + 7 = 0$

$\Rightarrow \frac{5(3-5\lambda)}{2}-\frac{2(2\lambda-4)}{2}+\frac{4(2-4\lambda)}{2}+7=0$

$\Rightarrow -45\lambda + 45 = 0 \Rightarrow \lambda = 1$

and the coordinates of the centre are $(-1, -1, -1)$.

II. Equation of the line passing through centre of sphere $(1, 2, -1)$ and perpendicular to the plane of sphere having direction ratios $(2, -2, 1)$ is



$s: x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$

$\frac{x-1}{2}=\frac{y-2}{-2}=\frac{z+1}{1}=\lambda$

Any point on the line is $P(2\lambda + 1, -2\lambda + 2, \lambda - 1)$

This point will lie in the plane if it satisfies the given equation of the plane such that

$2(2\lambda + 1) - 2(-2\lambda + 2) + (\lambda - 1) + 12 = 0$

$\Rightarrow 9\lambda + 9 = 0$

$\therefore \lambda = -1$

Hence, coordinates of point of contact are

$P(-1, 4, -2)$

43. II Let $P(u, v, w)$ be the foot of the perpendicular from the origin to the plane, then OP is normal to the plane, so that direction ratios of the normal to the plane are u, v, w and as it passes through the point $(2, -4, 6)$, its equations is $u(x - 2) + v(y + 4) + w(z - 6) = 0$.

Since (u, v, w) lies on it.

$u(u - 2) + v(v + 4) + w(w - 6) = 0$

$\Rightarrow u^2 + v^2 + w^2 - 2u + 4v - 6w = 0$

The locus of (u, v, w) is

$$x^2 + y^2 + z^2 - 2x + 4y - 6z = 0.$$

which is a sphere of radius $=\sqrt{1+4+9}=\sqrt{14}$.

III. The given line is

$\mathbf{L} : \mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda + (\mathbf{i} - \mathbf{j} + 4\mathbf{k})$

[From: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$]

where $\mathbf{a} = 2\mathbf{i} + 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$

and the plane is

$\pi : \mathbf{r} \cdot (\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = 5$ [From $\mathbf{r} \cdot \mathbf{n} = d$]

where $\mathbf{n} = \mathbf{i} + 5\mathbf{j} + \mathbf{k}$.

Now, $\mathbf{b} \cdot \mathbf{n} = (\mathbf{i} - \mathbf{j} + 4\mathbf{k}) \cdot (\mathbf{i} + 5\mathbf{j} + \mathbf{k})$

$= (1)(1) + (-1)(5) + (4)(1)$

$= 1 - 5 + 4 = 0.$

∴ The line L is perpendicular to the normal to plane π.

In the equation of the plane, replacing **r** by the position vector **a** of the point A on line L, we get

$\mathbf{a} \cdot \mathbf{n} = 5$

$\Rightarrow (2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = 5$

$\Rightarrow (2)(1) + (-2)(5) + (3)(1) = 5$

$\Rightarrow 2 - 10 + 3 = 5 \Rightarrow -5 = 5$ which is false.

Thus, A does not lie on plane π.

Hence, the line L is parallel to plane π and does not lie on plane π.

Distance of line L from plane π

= Distance of any point on L from plane π

= Distance of A from plane π

$$= \frac{|\mathbf{a} \cdot \mathbf{n} - 5|}{|\mathbf{n}|} = \frac{(2\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (\hat{i} + 5\hat{j} + \hat{k}) - 5}{\sqrt{(1)^2 + (5)^2 + 1)^2}}$$

$$= \frac{|2 - 10 + 3 - 5|}{\sqrt{27}} = \frac{10}{3\sqrt{3}}.$$

Hence, the given line L is parallel to the plane π and the distance between them is $\frac{10}{3\sqrt{3}}$.

IV. Let the equation of plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad (1)$$

The plane meets the axes in A, B and C, the coordinates of A, B and C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively. Coordinates of the centroid of ΔABC are

$$\left(\frac{a+0+0}{3}, \frac{0+b+0}{3}, \frac{0+0+c}{3}\right) = \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right).$$

But the given coordinates of the centroid of ΔABC are (α, β, γ)

$\therefore \frac{a}{3} = \alpha \Rightarrow a = 3\alpha$

$\frac{b}{3} = \beta \Rightarrow b = 3\beta$

$\frac{c}{3} = \gamma \Rightarrow c = 3\gamma$

Substituting these values of a, b, c in (1), we get

$$\frac{x}{3\alpha} + \frac{y}{3\beta} + \frac{z}{3\gamma} = 1$$

$$\Rightarrow \frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$$

Previous Year's Questions

44. Key Idea: A line will be a plane, iff

(a) the normal to the plane is perpendicular to the line.

(b) a point on the line lies on the plane.

In the given choices equation $x - y + z = 1$ is satisfied by (3, 2, 0) and (4, 7, 4). Thus the equation of plane which passes throught (3, 2, 0) and the line. $\frac{x-4}{1} = \frac{y-7}{5} = \frac{z-4}{4}$ has equation $x - y + z = 1$.

45. A parallelopiped is formed by planes drawn through the points (2, 3, 5) and (5, 9, 7), parallel to the coordinate planes. Let a, b, c be the lengths of edges, then

$a = 5 - 2 = 3$, $b = 9 - 3 = 6$, and $c = 7 - 5 = 2$

So the length of diagonal of the parallelopiped

$= \sqrt{a^2 + b^2 + c^2}$

$= \sqrt{9 + 36 + 4}$

$= \sqrt{49} = 7$ unit

46. Key Idea : A line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ lie in the plane $ax + by + cz + d = 0$, if

(a) $al + bm + cn = 0$, and

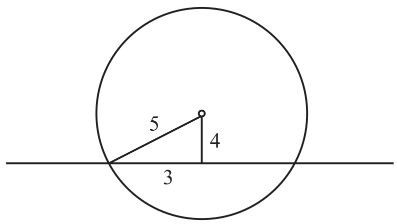
(b) $ax_1 + by_1 + cz_1 + d = 0$.

Therefore, the equation of the plane containing the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$. If

$al + bm + cn = 0$

47.



48. $1 = \cos\theta$, $m = \cos\theta$, $n = \cos\beta$

$\cos^2\theta + \cos^2\theta + \cos^2\beta = 1 \Rightarrow 2\cos^2\theta = 1 - \cos^2\beta = \sin^2\beta = 3\sin^2\theta$ (given) .

And so $\cos^2\theta = 3/5$.

49. Given equation of planes:

$2x + y + 2z - 8 = 0$, $4x + 2y + 4z + 5 = 0 \Rightarrow 2x + y + 2z + 5/2 = 0$

Hence the distance between planes

$$= \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|-8 - 5/2|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{7}{2}.$$

50. Any point on the line $\frac{x}{1}=\frac{y+a}{1}=\frac{z}{1}=t_1$ (say) is (t_1, $t_1 - a$, t_1) and any point on the line $\frac{x+a}{2}=\frac{y}{1}=\frac{z}{1}=t_2$ (say) is ($2t_2 - a$, t_2, t_2).

Now direction cosine of the lines intersecting the above lines is proportional to

$(2t_2 - a - t_1, t_2 - t_1 + a, t_2 - t_1)$

Hence $(2t_2 - a - t_1 = 2k$, $t_2 - t_1 + a = k$ and $t_2 - t_1 = 2k$

On solving these equations, we get $t_1 = 3a$, $t_2 = a$.

Hence the points are $(3a, 2a, 3a)$ and (a, a, a).

51. Given lines $\frac{x-1}{1}=\frac{y+3}{-\lambda}=\frac{z-1}{\lambda}=s$ and $\frac{x}{1/2}=\frac{y-1}{1}=\frac{z-2}{-1}=t$ are coplanar then plane passing through these lines has normal perpendicular to these lines

$\Rightarrow$ $a - b\lambda + c\lambda = 0$ and $\frac{a}{2}+b-c=0$ (where a, b, c are direction ratios of the normal to the plan)

From above equations, we get $\lambda = -2$.

52. Required plane is $S_1 - S_2 = 0$

where $S_1 = x^2 + y^2 + z^2 + 7x - 2y - z - 13 = 0$ and

$S_2 = x^2 + y^2 + z^2 - 3x + 3y + 4z - 8 = 0$

which implies that $2x - y - z = 1$.

53. Angle between line and normal to plane is

$\cos\left(\frac{\pi}{2}-\theta\right)=\frac{2-2+2\sqrt{\lambda}}{3\times\sqrt{5+\lambda}}$ where θ is angle between line & plane

$$\Rightarrow \sin\theta=\frac{2\sqrt{\lambda}}{3\sqrt{5+\lambda}}=\frac{1}{3}$$

$$\Rightarrow \lambda=\frac{5}{3}.$$

54. Plane

$2ax - 3ay + 4az + 6 = 0$ passes through the mid point of the centre of spheres $x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$ and $x^2 + y^2 + z^2 - 10x + 4y - 2z = 8$ respectively centre of spheres are $(-3, 4, 1)$ and $(5, -2, 1)$

Mid point of centre is (1, 1, 1)

Satisfying this in the equation of plane, we get

$2a - 3a + 4a + 6 = 0$

$\Rightarrow$ $a = -2$.

55. Equation of lines $\frac{x-b}{a}=y=\frac{z-d}{c}$

and, $\frac{x-b'}{a'}=y=\frac{z-d'}{c'}$

Lines are perpendicular $\Rightarrow aa' + 1 + cc' = 0$.

56. If (α, β, γ) be the image, then

$$\frac{a-1}{2}-2\left(\frac{\beta+3}{2}\right)=0$$

$\therefore$ $\alpha - 1 - 2\beta - 6 \Rightarrow \alpha - 2\beta = 7$ (1)

and $\frac{\alpha+1}{1}=\frac{\beta-3}{-2}=\frac{\gamma-4}{0}$ (2)

From (1) and (2)

$$\alpha=\frac{9}{5}, \beta=-\frac{13}{5}, \gamma=4$$

No option matches.

57. If direction cosines of L be l, m, n, then $2l + 3m + n = 0$

Also, $l + 3m + 2n = 0$

Solving both the equations, we get

$$\frac{l}{3}=\frac{m}{-3}=\frac{n}{3}$$

$$\therefore\ l:m:n=\frac{1}{\sqrt{3}}:-\frac{1}{\sqrt{3}}:\frac{1}{\sqrt{3}} \Rightarrow \cos\alpha=\frac{1}{\sqrt{3}}$$

58. Given that $l=\cos\frac{\pi}{4}, m=\cos\frac{\pi}{4}$

We know $l^2 + m^2 + n^2 = 1$

$$\Rightarrow \frac{1}{2}+\frac{1}{2}+n^2=1$$

$\Rightarrow$ $n = 0$

Hence, the angle with positive direction of z-axis is $\frac{\pi}{2}$.

59. Coordinates of centre are (3, 6, 1)

Let the coordinates of the other end of diameter (α, β, γ).

Then, $\frac{\alpha+2}{2}=3, \frac{\beta+3}{2}=6, \frac{\gamma+5}{2}=1$

Hence, $\alpha = 4$, $\beta = 9$ and $\gamma = -3$.

60. Equation of line passing through $(5, 1, a)$ and $(3, b, 1)$ is

$$\frac{x-5}{2}=\frac{y-1}{1-b}=\frac{z-a}{a-1}=\lambda$$

If line crosses yz-plane i.e., $x = 0$.

Then $x = 2\lambda + 5 = 0$

$\Rightarrow$ $\lambda = -5/2$,

Since, $y=\lambda(1-b)+1=\frac{17}{2}$

$$\Rightarrow -\frac{5}{2}(1-b)+1=\frac{17}{2}$$

$\Rightarrow$ $b = 4$.

Also, $z=\lambda(a-1)+a=-\frac{13}{2}$

$$\Rightarrow -\frac{5}{2}(a-1)+a=-\frac{13}{2}$$

$\Rightarrow$ $a = 6$

61. Given lines $\frac{x-1}{k}=\frac{y-2}{2}=\frac{z-3}{3}$ and $\frac{x-2}{3}=\frac{y-3}{k}=\frac{z-1}{2}$

Since lines intersect in a point, we have

$$\begin{vmatrix} k & 2 & 3 \\ 3 & k & 2 \\ 1 & 1 & -2 \end{vmatrix}=0$$

$\therefore$ $2k^2 + 5k - 25 = 0$ and so, $k = -5$ or $5/2$.

62. Direction ratios of line = (3, −5, 2)
Direction ratios of normal to the plane = (1, 3, $-\alpha$)
Since line is perpendicular to normal we write
$3(1) - 5(3) + 2(-\alpha) = 0$
$\Rightarrow 3 - 15 - 2\alpha = 0 \Rightarrow 2\alpha = -12 \Rightarrow \alpha = -6$
Also (2, 1, −2) lies on the plane
$2 + 3 + 6(-2) + \beta = 0 \Rightarrow \beta = 7$
$\therefore (\alpha, \beta) = (-6, 7)$

63. $l = \cos 45° = \frac{1}{\sqrt{2}}$

$m = \cos 120° = -\frac{1}{2}$

$n = \cos\theta$

where θ is the angle which line makes with positive z-axis.
Now $l^2 + m^2 + n^2 = 1$

$\Rightarrow \frac{1}{2} + \frac{1}{4} + \cos^2\theta = 1$

$\cos^2\theta = \frac{1}{4}$

$\Rightarrow \cos\theta = \frac{1}{2}$ (θ being acute)

$\Rightarrow \theta = \frac{\pi}{3}$

64. $\cos\theta = \sqrt{\frac{5}{14}}$

$\sin\theta = \sqrt{\frac{3}{14}}$

$\sin\theta = \frac{1+4+3\lambda}{\sqrt{1+4+\lambda^2}\sqrt{1+4+9}}$

$\frac{3}{\sqrt{14}} = \frac{5+3\lambda}{\sqrt{5+\lambda^2}\sqrt{14}} \Rightarrow \lambda = \frac{2}{3}$

65. Statement - 1: AB is perpendicular to given line and mid point of AB lies on line
Statement - 2 is true but it is not correct explanation as it is bisector only.
If it is perpendicular bisector then only statement - 2 is correct explanation.

66. Equation of plane parallel to $x - 2y + 2z - 5 = 0$ is $x - 2y + 2z + k = 0$ (1)
Perpendicular distance from $O(0, 0, 0)$ to (1) is I

$\frac{|k|}{\sqrt{1+4+4}} = 1$

$\Rightarrow |k| = 3$
$\Rightarrow k = \pm 3$
$\therefore x - 2y + 2z - 3 = 0$

67. Any point on $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4} = t$ is $(2t + 1, 3t - 1, 4t + 1)$
And any point on $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1} = s$ is $(s + 3, 2s + k, s)$
Given lines are intersecting
$\Rightarrow t = -\frac{3}{2}$ and $s = -5 \quad \therefore \quad k = \frac{9}{2}$

68. Since, $\begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$

$1(1 + 2k) + 1(1 + k^2) - 1(2 - k) = 0$
$k^2 + 1 + 2k + 1 - 2 + k = 0$
$k^2 + 3k = 0$
$(k)(k + 3) = 0$
$\therefore$ 2 values of k.

69. Given $4x + 2y + 4z = 16$
$4x + 2y + 4z = -5$

$\therefore d_{\min} = \frac{21}{\sqrt{36}} = \frac{21}{6} = \frac{7}{2}$.

70. Line is parallel to plane
Image of (1, 3, 4) is (−3, 5, 2).

71. $l = -m - n$
$m^2 + n^2 = (m + n)^2$
$\Rightarrow mn = 0$
So possibilities are $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ or $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$.

72. Let the point of intersection be $(2 + 3\lambda, 4\lambda - 1, 12\lambda + 2)$
$(2 + 3\lambda) - (4\lambda - 1) + 12\lambda + 2 = 16$
$\Rightarrow 11\lambda = 11$
$\Rightarrow \lambda = 1$
$\Rightarrow$ point of intersection is (5, 3, 14)
$\Rightarrow$ distance $= \sqrt{(5-1)^2 + 9 + 12^2}$
$= \sqrt{16 + 9 + 144} = 13$.

73. Let equation of plane be $(2x - 5y + z - 3) + \lambda(x + y + 4z - 5) = 0$
As plane is parallel to $x + 3y + 6z - 1 = 0$

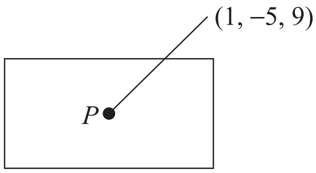
$\therefore \frac{2+\lambda}{1} = \frac{\lambda - 5}{3} = \frac{1+4\lambda}{6}$

$\Rightarrow 6 + 3\lambda = \lambda - 5$
$\Rightarrow 11 = -2\lambda$
$\Rightarrow \lambda = -\frac{11}{2}$

Also, $6\lambda - 30 = 3 + 12\lambda$
$\Rightarrow -6\lambda = 33$
So the equation of required plane is
$(4x - 10y + 2z - 6) - 11(x + y + 4z - 5) = 0$
$\Rightarrow -7x - 21y - 42z + 49 = 0$
$\Rightarrow x + 3y + 6z - 7 = 0$.

74. Equation of line parallel to $x = y = z$ through $(1, -5, 9)$ is $\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = \lambda$

If $P(\lambda + 1, \lambda - 5, \lambda + 9)$ be the point of intersection of line and plane, we have



$\lambda + 1 - \lambda + 5 + \lambda + 9 = 5$

$\Rightarrow \quad \lambda = -10$

$\Rightarrow$ Coordinates of the point P are $(-9, -15, -1)$

$\therefore$ Required distance $= 10\sqrt{3}$

75. Given line is

$\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$

and given plane is $lx + my - z = 9$

Now, it is given that the line lies on the plane

$\therefore \quad 2l - m - 3 = 0 \Rightarrow 2l - m = 3$ (1)

Also, $(3, -2, -4)$ lies on the plane

$\therefore \quad 3l - 2m = 5$ (2)

Solving (1) and (2), we get

$l = 1, m = -1$

$\therefore \quad l^2 + m^2 = 2$